Characterization of a Linear Program in Standard Form by a Family of Linear Programs with Inequality Constraints

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In the linear programming theory, conditions for feasibility, unboundedness or existence of an optimal solution of a linear program in standard form

$$\text{(P)} \qquad \min\{c^{\mathrm{T}}x;\ Ax = b,\ x \geqq 0\}$$

are given in terms of the *dual* problem. It is the purpose of this short note to show that such conditions can be also given in terms of a family of *primal* problems which arise when releasing each equality constraint in (P) by replacing it by either of the two associated inequality constraints. This result is almost certainly of no practical value, but may be of some theoretical interest.

We shall first formulate an auxiliary result concerning the existence of a solution of a system of linear equations

$$\text{(S)} \qquad Ax = b$$

with an $m \times n$ matrix A. Denote $Y = \{y \in R^m;\ |y_j| = 1 \text{ for } j = 1, \ldots, m\}$, so that Y consists of 2^m elements, and for each $y \in Y$ denote $D_y = \operatorname{diag}\{y_1, \ldots, y_m\}$, the diagonal matrix with diagonal vector y. Together with (S), we shall consider the family of systems of linear inequalities of the form

$$(\mathrm{S}_y) \qquad D_y A x \leqq D_y b$$

for all $y \in Y$. It is easy to see that the i-th row in (S_y) has the form $(Ax)_i \leqq b_i$ if $y_i = 1$ and the form $-(Ax)_i \leqq -b_i$ if $y_i = -1$.

Now we have this

Lemma. Let for each $y \in Y$ the system (S_y) have a solution x_y. Then (S) has a solution which is a convex combination of the x_y's.

Proof. We shall prove that the system of linear equations

$$(1) \qquad \sum_{y \in Y} \lambda_y (Ax_y) = b, \qquad \sum_{y \in Y} \lambda_y = 1$$

has a nonnegative solution $\lambda_y \in R^1$, $y \in Y$. In the light of the Farkas lemma ([1]), this amounts to proving that for each $p \in R^m$ and $p_0 \in R^1$, if $p^{\mathrm{T}}Ax_y + p_0 \geqq 0$ for each $y \in Y$, then $p^{\mathrm{T}}b + p_0 \geqq 0$. So let p, p_0 satisfy $p^{\mathrm{T}}Ax_y + p_0 \geqq 0$ for each $y \in Y$. Define a vector $z \in Y$ in the following way: $z_i = 1$ if $p_i \geqq 0$ and $z_i = -1$ if $p_i < 0$ $(i = 1, \ldots, m)$. Since x_z satisfies $(Ax_z)_i \leqq b_i$ if $z_i = 1$ and $(Ax_z)_i \geqq b_i$ if $z_i = -1$, we have that $p^{\mathrm{T}}b + p_0 \geqq p^{\mathrm{T}}Ax_z + p_0 \geqq 0$, and we are done. So the system (1) has a nonnegative solution. Taking $x = \sum_{y \in Y} \lambda_y x_y$, we have that $Ax = b$ and x is a convex combination of the x_y's, which concludes the proof. □

Let us now return to our original linear program (P). We shall show in the next theorem that (P) can be characterized in terms of the family of linear programs of the form

$$(\mathrm{P}_y) \qquad \min \{c^{\mathrm{T}}x;\ D_y Ax \leqq D_y b,\ x \geqq 0\}$$

for all $y \in Y$ (so that there are 2^m of them). Notice that the characterization is given purely in primal terms.

Theorem. The following assertions hold for (P) and (P_y), $y \in Y$:

(i) (P) is feasible if and only if (P_y) is feasible for each $y \in Y$,

(ii) (P) is unbounded if and only if (P_y) is unbounded for each $y \in Y$,

(iii) (P) has a finite optimum if and only if each (P_y), $y \in Y$, is feasible and at least one (P_y) has a finite optimum,

(iv) if x^* is an optimal solution to (P) and x_y^* an optimal solution to some (P_y), then $c^{\mathrm{T}}x^* \geqq c^{\mathrm{T}}x_y^*$,

(v) if (P) has a finite optimum, then there exists at least one $y \in Y$ such that each optimal solution of (P) is also an optimal solution of (P_y), so that the optimal values of (P) and (P_y) are equal,

(vi) if x is a feasible solution of (P) satisfying $c^{\mathrm{T}}x = c^{\mathrm{T}}x_y^*$ for an optimal solution x_y^* of some problem (P_y), $y \in Y$, then x is an optimal solution to (P).

Remark. The assertions (iv), (v), (vi) may be considered analogues to weak duality, duality theorem and optimality conditions of LP, respectively.

Proof. (i) If (P) has a feasible solution x, then x is also feasible for each (P_y), $y \in Y$. Conversely, if each (P_y) has a feasible solution x_y, then our Lemma gives that $Ax = b$ has a solution which is a convex combination of the x_y's, hence nonnegative, thus feasible for (P).

(ii) If (P) is unbounded, then also each (P_y), $y \in Y$, is unbounded since each feasible solution of (P) is also feasible for each (P_y). Conversely, assume that each (P_y) is unbounded. Then, as well-known ([1]), for each (P_y) there exists a vector x_y such that $D_y A x_y \leqq 0$, $x_y \geqq 0$, $c^{\mathrm{T}}x_y < 0$. Using the Lemma, we get that there exists an x_0 such that $Ax_0 = 0$ and x_0 is a convex combination of the x_y's, so that $x_0 \geqq 0$ and $c^{\mathrm{T}}x_0 < 0$. Then for each feasible solution x_1 of (P), the whole half-ray $\{x_1 + \mu x_0;\ \mu \geqq 0\}$ consists of feasible solutions for (P) and the objective decreases to $-\infty$ along it, so (P) is unbounded.

(iii) If (P) has a finite optimum, then each (P_y) is feasible due to (i) and at least one (P_y) must have a finite optimum since otherwise (P) would be unbounded due to (ii). Conversely, if each (P_y) is feasible and some (P_z) has a finite optimum, then (P) is feasible due to (i) and its objective is bounded from below by the optimal value of (P_z), so that (P) has a finite optimum.

(iv) The assertion follows obviously from the fact that the set of feasible solutions of (P) is a part of the set of feasible solutions of (P_y).

(v) Let (P) have a finite optimum. Take an optimal solution p^* of its dual problem

$$\text{(D)} \qquad \max \{b^{\mathrm{T}}p;\ A^{\mathrm{T}}p \leqq c\}\,,$$

so that $x^{*\mathrm{T}}(A^{\mathrm{T}}p^* - c) = 0$ for any optimal solution x^* of (P). Define y as follows: $y_i = 1$ if $p_i^* \leqq 0$ and $y_i = -1$ otherwise. Now, consider (P_y) and its dual problem

$$(\overline{\mathrm{D}}_y) \qquad \max \{b^{\mathrm{T}}(-D_y\pi);\ A^{\mathrm{T}}(-D_y\pi) \leqq c,\ \pi \geqq 0\}\,.$$

Obviously, x^* is feasible for (P_y) and $\pi^* = -D_yp^*$ is feasible for $(\overline{\mathrm{D}}_y)$ and, moreover, $x^{*\mathrm{T}}(A^{\mathrm{T}}(-D_y\pi^*) - c) = x^{*\mathrm{T}}(A^{\mathrm{T}}p^* - c) = 0$, $\pi^{*\mathrm{T}}(D_yAx^* - D_yb) = 0$, so that the optimality conditions for the pair (P_y), $(\overline{\mathrm{D}}_y)$ are satisfied, showing that x^* is an optimal solution of (P_y).

(vi) Since each feasible solution $\tilde{x}$ of (P) is also feasible for (P_y), we have $c^{\mathrm{T}}\tilde{x} \geqq c^{\mathrm{T}}x_y^* = c^{\mathrm{T}}x$, which means that x is an optimal solution to (P). □

If (P) has a finite optimum, then at least one (P_y) has a finite optimum (assertion (iii)), but there may exist another (P_y)'s which are unbounded. Consider a simple example

$$\min x_1$$

subject to

$$x_1 + x_2 = 0$$
$$x_1 - x_2 = 0$$
$$x_1 \geqq 0\,, \quad x_2 \geqq 0$$

which has a unique feasible solution $(0, 0)^{\mathrm{T}}$. For $y = (1, 1)^{\mathrm{T}}$, the set of feasible solutions of (P_y) contains the half-ray $\{(x_1, 0)^{\mathrm{T}};\ x_1 \leqq 0\}$, hence (P_y) is unbounded.

Reference

[1] Murty, K. G.: Linear and Combinatorial Programming. Wiley, New York 1976.

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Resumé

CHARAKTERIZACE LINEÁRNÍHO PROGRAMU VE STANDARDNÍ FORMĚ SOUBOREM LINEÁRNÍCH PROGRAMŮ S OMEZENÍMI VE TVARU NEROVNOSTÍ

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V článku je ukázáno, že úloha lineárního programování s omezeními ve tvaru rovností může být charakterizována (co se týče přípustnosti, neomezenosti, existence optimálního řešení) souborem úloh lineárního programování, které vzniknou nahrazením každé rovnice v soustavě omezení jednou ze dvou odpovídajících nerovností.