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Anschrift: Dr. Peter Paul Klein Rechenzentrum der TU Clausthal Erzstraße 51 D-3392 Clausthal-Zellerfeld, BRD

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Rohn, J.

Real Eigenvalues of an Interval Matrix with Rank One Radius

In this paper, we are concerned with eigenvalues and eigenvectors of an $n \times n$ interval matrix $A^I = \{A; A_c - \Delta \leqq A \leqq \leqq A_c + \Delta\}$, where the radius Δ is of the form $\Delta = qp^T$, q and p being column vectors, $q \geqq 0$, $p > 0$. As it will be seen, this special case allows for rather explicit results. We will be interested only in real eigenvalues since the complex case seemingly cannot be handled by the method used.

For an interval matrix in the above form we shall consider the set of real eigenvalues

$$L = \{\lambda \in R^1; Ax = \lambda x, A \in A^I, x \neq 0\}$$

and for each $\lambda \in R^1$ the set of eigenvectors

$$X_\lambda = \{x; Ax = \lambda x, A \in A^I, x \neq 0\}.$$

We shall first give a description of the X_λ's, then, using the fact that $\lambda \in L$ iff $X_\lambda \neq \emptyset$, we shall derive a necessary and sufficient condition for a given $\lambda \in R^1$ to belong to L. Notation: with the above vector $p > 0$ we associate the norm $\|x\|_p = \sum_i p_i|x_i|$, and for a vector $x = (x_i)$ we define its absolute value by $|x| = (|x_i|)$; let $e = (1, 1, \ldots, 1)^T$. For a $t \in R^n$, we denote by D_t the diagonal matrix with diagonal vector t; $E = D_e$ is the unit matrix.

Theorem 1: *Let $\lambda \in R^1$. Then there holds*

$$X_\lambda = \{x; -q \leqq (A_c - \lambda E)\frac{x}{\|x\|_p} \leqq q, x \neq 0\}. \tag{1}$$

Proof: Let $x \in X_\lambda$, i.e. $Ax = \lambda x$ for some $A \in A^I$ and $x \neq 0$. Then

$$|(A_c - \lambda E)x| = |(A - \lambda E)x + (A_c - A)x| \leqq |A_c - A| \cdot |x| \leqq qp^T|x| = \|x\|_p q,$$

hence $-q \leqq (A_c - \lambda E)\frac{x}{\|x\|_p} \leqq q$. Conversely, if the last inequality holds for some $x \neq 0$, then $|(A_c - \lambda E)x| \leqq qp^T|x|$. Define a vector $t \in R^n$ by

$$t_i = ((A_c - \lambda E)x)_i/(qp^T|x|)_i \quad \text{if} \quad (qp^T|x|)_i \neq 0 \quad \text{and} \quad t_i = 1 \quad \text{otherwise} \quad (i = 1, \ldots, n);$$

then $(A_c - \lambda E)x = D_t qp^T|x|$. Let z be the signature vector of x (i.e. $z_i = 1$ if $x_i \geqq 0$ and $z_i = -1$ if $x_i < 0$), then $|x| = D_z x$, so that $(A_c - D_t qp^T D_z)x = \lambda x$. Since $|t| \leqq e$, we have $|D_t qp^T D_z| \leqq qp^T$, hence $A_c - D_t qp^T D_z \in A^I$, implying $x \in X_\lambda$. ■

We shall now give a necessary and sufficient condition for a given λ to belong to L. Denote $Y = \{y \in R^n; |y| = e\}$, i.e. the set of all ± 1-vectors.

Theorem 2: *Let $\lambda \in R^1$ be not an eigenvalue of A_c. Then $\lambda \in L$ if and only if it satisfies*

$$\|(A_c - \lambda E)^{-1} D_q y\|_p \geqq 1 \tag{2}$$

for some $y \in Y$.

Proof: Let $\lambda \in L$ and let λ be not an eigenvalue of A_c. Then $X_\lambda \neq \emptyset$, hence in view of Theorem 1 there exists a vector x with $\|x\|_p = 1$ such that $(A_c - \lambda E)\,x = t$, $|t| \leqq q$. Let $\bar{y}$ be a vector from Y at which $\|(A_c - \lambda E)^{-1} D_q y\|_p$ achieves its maximum over Y. Then from the convexity of the norm we obtain

$$1 = \|x\|_p = \|(A_c - \lambda E)^{-1}\,t\|_p \leqq \|(A_c - \lambda E)^{-1} D_q \bar{y}\|_p ,$$

which proves (2). Conversely, let (2) hold. Put $x = (A_c - \lambda E)^{-1} D_q y$, then $|(A_c - \lambda E)\,x| = q$ and $\|x\|_p \geqq 1$, hence the vector $x_0 = x/\|x\|_p$ satisfies $|(A_c - \lambda E)\,x_0| \leqq q$, $\|x_0\|_p = 1$, so that Theorem 1 assures X_λ to be nonempty, which means that $\lambda \in L$. ■

This result shows that the test whether $\lambda \in L$ can be in principle performed by finite means, but actually the amount of computation needed may be too large since Y consists of 2^n elements. Nevertheless, we shall use this characterization for a full description of L under some additional assumptions (A1), (A2), (A3), which we shall first discuss in some detail. First we shall assume that

(A1) *Each $A \in A^I$ has m simple real eigenvalues $\lambda_1(A) < \lambda_2(A) < \ldots < \lambda_m(A)$, where m is constant over A^I.*

Based on this assumption, we may define the sets $L_i = \{\lambda_i(A);\ A \in A^I\}$, $i = 1, \ldots, m$. Obviously, $L = \bigcup_1^m L_i$. Next we shall assume the L_i's to satisfy

(A2) *$\bar{L}_i \cap L_j = \emptyset$ for each $i \neq j$.*

According to (A1), both the eigenspace and the left eigenspace of each $\lambda_i(A)$ are one-dimensional. Denote by $x_i(A)$ and $\pi_i(A)$ the unique eigenvector and left eigenvector, respectively, such that $\|x_i(A)\|_p = \|\pi_i(A)\|_p = 1$ and whose first nonzero entry is positive. We shall assume a signature constancy of these vectors:

(A3) *For each $i \in \{1, \ldots, m\}$ there exist z_i, $y_i \in Y$ such that $D_{z_i} x_i(A) > 0$, $D_{y_i} \pi_i(A) > 0$ for each $A \in A^I$.*

We shall now show that under these assumptions each L_i can be described explicitly.

Theorem 3: *Let $q > 0$, $p > 0$ and let* (A1), (A2), (A3) *hold. Then for each $i \in \{1, \ldots, m\}$ we have $L_i = [\underline{\lambda}_i, \bar{\lambda}_i]$, where*

$$\underline{\lambda}_i = \min\{\lambda_i(A_c - B_i),\, \lambda_i(A_c + B_i)\}\,, \qquad \bar{\lambda}_i = \max\{\lambda_i(A_c - B_i),\, \lambda_i(A_c + B_i)\}$$

with $B_i = D_{y_i} q p^{\mathrm{T}} D_{z_i}$.

Proof: Let $i \in \{1, \ldots, m\}$. From (A2) it follows that L_i is compact. Since $qp^{\mathrm{T}} > 0$, we have that $\lambda_i(A_c)$ belongs to L_i^0, the interior of L_i. Let $\lambda \in \partial L_i = L_i - L_i^0$. Then $\lambda \neq \lambda_i(A_c)$, hence according to Theorem 2 there exists a $y \in Y$ with $\|(A_c - \lambda E)^{-1} D_q y\|_p \geqq 1$. If this inequality were sharp, we would have $\lambda \in L_i^0$, a contradiction. Hence $\|(A_c - \lambda E)^{-1} \times D_q y\|_p = 1$. Denote $x = (A_c - \lambda E)^{-1} D_q y$ and let z be the signature vector of x. Then $p^{\mathrm{T}} D_z x = 1$ and $(A_c - \lambda E)\,x = D_q y = D_y q p^{\mathrm{T}} D_z x$, hence

$$(A_c - D_y q p^{\mathrm{T}} D_z)\,x = \lambda x\,, \tag{4}$$

which means that x is an eigenvector corresponding to λ, hence its signature vector z must be equal to z_i or $-z_i$ according to (A3). Further, let $\pi = (A_c^{\mathrm{T}} - \lambda E)^{-1} D_z p$, then $\pi^{\mathrm{T}} D_q y = \|(A_c - \lambda E)^{-1} D_q y\|_p = 1$. If there were $\pi_j y_j < 0$ for some j, then defining y^0 by $y_j^0 = -y_j$ and $y_k^0 = y_k$ otherwise, we would have $\|(A_c - \lambda E)^{-1} D_q y^0\|_p = \pi^{\mathrm{T}} D_q y^0 > 1$ implying $\lambda \in L_i^0$, a contradiction. Hence $\pi_j y_j \geqq 0$ for each j and from $(A_c^{\mathrm{T}} - \lambda E)\,\pi = D_z p = D_z p q^{\mathrm{T}} D_y \pi$ we obtain $(A_c - D_y q p^{\mathrm{T}} D_z)^{\mathrm{T}}\,\pi = \lambda \pi$, thus π is a left eigenvector and its signature vector y must be equal to y_i or $-y_i$. Thus we obtain from (4) that either $\lambda = \lambda_i(A_c - D_{y_i} q p^{\mathrm{T}} D_{z_i})$, or $\lambda = \lambda_i(A_c + D_{y_i} q p^{\mathrm{T}} D_{z_i})$. We have proved that L_i is compact, has a nonempty interior and two boundary points. Thus L_i is a compact interval whose endpoints are the two boundary points, which proves (3). ■

A special case of Theorem 3 for A^I symmetric ($p = q$) was proved in [2]. While working on this problem, I received the preprint [1] by A. Deif where Theorem 3 was proved under other assumptions and by another method; the results presented here were achieved independently of it.

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Address: Dr. Jiří Rohn, Faculty of Mathematics and Physics, Malostranské nám. 25, 11800 Prague, Czechoslovakia