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NONSINGULARITY AND P-MATRICES

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Summary. New proofs of two previously published theorems relating nonsingularity of interval matrices to P-matrices are given.

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In [5] we proved, in a broader frame of the problem of solving linear interval systems, two theorems relating nonsingularity of interval matrices to P-matrices (Theorems 1 and 2 below). It is the purpose of this paper to give alternative proofs of them, from which it can be perhaps better seen how nonsingularity is intertwinned with P-property. We also include some consequences implied by the properties of P-matrices.

We begin with this simple auxiliary result:

Lemma. *Let A be a nonsingular $n \times n$ matrix and let B be an $n \times n$ matrix whose rows, except the j-th, are zero. Let $1 + (BA^{-1})_{jj} \leqq 0$. Then there exists a $t \in (0, 1]$ such that $A + tB$ is singular.*

Proof. Consider the function φ of one real variable defined by $\varphi(\tau) = 1 + \tau(BA^{-1})_{jj}$. Since $\varphi(0) > 0$ and $\varphi(1) \leqq 0$, there exists a $t \in (0, 1]$ such that $\varphi(t) = 0$. Then the matrix $A + tB = (E + tBA^{-1}) A$ is singular since $\det (E + tBA^{-1}) = 1 + t(BA^{-1})_{jj} = 0$. ∎

Let A^-, A^+ be two $n \times n$ matrices, $A^- \leqq A^+$ (the inequality to be understood componentwise). The set of matrices

$$A^I = \{A; A^- \leqq A \leqq A^+\}$$

is called an interval matrix; we say that A^I is nonsingular (in [5]: regular) if each $A \in A^I$ is nonsingular. A square matrix A is said to be a P-matrix [1] if all its principal minors are positive.

First, we have this result:

Theorem 1. *Let A^I be nonsingular. Then for each $A_1, A_2 \in A^I$, both $A_1 A_2^{-1}$ and $A_1^{-1} A_2$ are P-matrices.*

Proof. The proof consists of several steps. Let $A_1, A_2 \in A^I$.

(1) We shall prove that all *leading* principal minors $m_1, \ldots, m_n$ of $A_1 A_2^{-1}$ are positive. Put $D = A_1 - A_2$ so that $A_1 A_2^{-1} = E + D A_2^{-1}$, and denote by D_j $(j = 1, \ldots, n)$ the matrix whose first j rows are identical with those of D and the remaining ones are zero. Then

$$m_j = \det(E + D_j A_2^{-1})$$

holds for $j = 1, \ldots, n$. We shall prove by induction that $m_j > 0$ for each j:

(1.1) Case $j = 1$. Since $m_1 = \det(E + D_1 A_2^{-1}) = 1 + (D_1 A_2^{-1})_{11}$, the above lemma implies $m_1 > 0$, for otherwise the matrix $A_2 + tD_1$ would be singular for some $t \in (0, 1]$ but $A_2 + tD_1 \in A^I$, which is a contradiction.

(1.2) Case $j > 1$. Assume that $m_{j-1} > 0$ and consider the matrix

$$(E + D_j A_2^{-1})(E + D_{j-1} A_2^{-1})^{-1} = E + (D_j - D_{j-1})(A_2 + D_{j-1})^{-1}.$$

Taking determinants on both sides we obtain

$$\frac{m_j}{m_{j-1}} = 1 + [(D_j - D_{j-1})(A_2 + D_{j-1})^{-1}]_{jj}.$$

If the right-hand side were nonpositive, then, according to the lemma, $A_2 + D_{j-1} + t(D_j - D_{j-1})$ would be singular for some $t \in (0, 1]$, which is a contradiction since it is a matrix from A^I. Hence

$$\frac{m_j}{m_{j-1}} > 0$$

holds, which in conjunction with the induction hypothesis gives that $m_j > 0$, which concludes the inductive proof.

(2) Second we shall prove that each principal minor of $A_1 A_2^{-1}$ is positive. Consider a principal minor formed from the rows and columns with indices $k_1, \ldots, k_r$, $1 \leqq r \leqq n$. Let R be any permutation matrix with $R_{k_j j} = 1$ $(j = 1, \ldots, r)$. Then the above minor is equal to the r-th leading principal minor of $R^T A_1 A_2^{-1} R = (R^T A_1 R) \cdot (R^T A_2 R)^{-1}$. Since the interval matrix $\{R^T A R;\ A \in A^I\}$ is nonsingular, all leading principal minors of $(R^T A_1 R)(R^T A_2 R)^{-1}$ are positive due to (1).

(3) To prove that $A_1^{-1} A_2$ is also a P-matrix, consider the transpose interval matrix $(A^I)^T = \{A^T;\ A \in A^I\}$. According to part (2), its nonsingularity implies that $(A_2^T)(A_1^T)^{-1} = (A_1^{-1} A_2)^T$ is a P-matrix, hence so is $A_1^{-1} A_2$. This completes the proof. ∎

We shall now show that the result can be in a certain sense reversed, so that the P-property of a finite number of matrices of the form $A_1^{-1}A_2$ will imply nonsingularity of A^I. To this end, let us denote

$$A_c = \tfrac{1}{2}(A^- + A^+)\,,$$
$$\Delta = \tfrac{1}{2}(A^+ - A^-)\,,$$

then $A^- = A_c - \Delta$, $A^+ = A_c + \Delta$, $\Delta \geqq 0$. A diagonal matrix S satisfying $|S_{ii}| = 1$ for each i is called a signature matrix, so that there are 2^n signature matrices of size n.

Theorem 2. *An interval matrix A^I is nonsingular if and only if for each signature matrix S, $A_c - S\Delta$ is nonsingular and $(A_c - S\Delta)^{-1}(A_c + S\Delta)$ is a P-matrix.*

Proof. The "only if" part being an obvious consequence of Theorem 1, we must prove the "if" part only. This will be done if we prove that for each $A \in A^I$ and each $b \in R^n$, the system of linear equations

$$Ax = b$$

has a solution, which, according to a theorem proved in [6], is equivalent to the fact that for each signature matrix S, the system of linear inequalities

$$(*) \qquad SAx \geqq Sb$$

has a solution. To show this, consider the linear complementarity problem

$$x_1 = (A_c - S\Delta)^{-1}(A_c + S\Delta)\,x_2 + (A_c - S\Delta)^{-1}\,b\,,$$
$$x_1^T x_2 = 0\,,$$
$$x_1 \geqq 0\,, \quad x_2 \geqq 0\,.$$

Since $(A_c - S\Delta)^{-1}(A_c + S\Delta)$ is a P-matrix by the assumption, this problem has a solution x_1, x_2, as proved in [7]. Then

$$A_c(x_1 - x_2) - S\Delta(x_1 + x_2) = b$$

and for each $A \in A^I$ we have

$$SA(x_1 - x_2) = SA_c(x_1 - x_2) + S(A - A_c)(x_1 - x_2) \geqq$$
$$\geqq SA_c(x_1 - x_2) - \Delta(x_1 + x_2) = Sb\,,$$

hence $(*)$ has a solution, which by virtue of the above-quoted theorem proves that A^I is nonsingular. ■

It is worth noting that the matrices $(A_c - S\Delta)^{-1}(A_c + S\Delta)$ cannot be replaced by matrices of the type $(A_c - S\Delta)(A_c + S\Delta)^{-1}$ in the formulation of Theorem 2:

Example 1 (communicated to the author by M. Baumann). Let

$$A^- = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}, \quad A^+ = \begin{pmatrix} 7 & 3 \\ 5 & 7 \end{pmatrix}.$$

Then $(A_c - S\Delta)(A_c + S\Delta)^{-1}$ is a P-matrix for each signature matrix S, but A^I contains the singular matrix

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}.$$

Since each positive definite (not necessarily symmetric) matrix is a P-matrix [1], we obtain a consequence:

Corollary 1. *For each signature matrix S, let $A_c - S\Delta$ be nonsingular and $(A_c - S\Delta)^{-1}(A_c + S\Delta)$ positive definite. Then A^I is nonsingular.*

The converse implication is, however, not true:

Example 2. Let

$$A^- = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^+ = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then A^I is obviously nonsingular, but none of the matrices $(A_c - S\Delta)^{-1}(A_c + S\Delta)$ is positive definite.

Finally, using the well-known properties of P-matrices, we may draw some consequences regarding nonsingular interval matrices:

Corollary 2. *Let A^I be nonsingular. Then for each $A_1, A_2 \in A^I$ we have:*

(i) *each diagonal element of both $A_1^{-1}A_2$ and $A_1A_2^{-1}$ is positive,*

(ii) *for each signature matrix S there exist x_1, x_2 such that $A_1x_1 = A_2x_2$, $Sx_1 > 0$, $Sx_2 > 0$,*

(iii) *for each signature matrix S there exist x_1, x_2 such that $A_1^{-1}x_1 = A_2^{-1}x_2$, $Sx_1 > 0$, $Sx_2 > 0$,*

(iv) *if $A_1x_1 = A_2x_2$ for some $x_1 \neq 0$, $x_2 \neq 0$, then $(x_1)_i(x_2)_i > 0$ for some $i \in \{1, \ldots, n\}$,*

(v) *if $A_1^{-1}x_1 = A_2^{-1}x_2$ for some $x_1 \neq 0$, $x_2 \neq 0$, then $(x_1)_i(x_2)_i > 0$ for some $i \in \{1, \ldots, n\}$.*

Proof. (i) follows from the fact that each diagonal element (i.e., first order minor) of a P-matrix is positive. (ii) Let S be a signature matrix. Then the interval matrix $\{AS;\ A \in A^I\}$ is nonsingular, hence $(A_1S)^{-1}(A_2S) = SA_1^{-1}A_2S$ is a P-matrix; then, as proved by Gale and Nikaido [3], there exists a $y_2 > 0$ such that $y_1 = SA_1^{-1}A_2Sy_2 > 0$. Setting $x_1 = Sy_1$, $x_2 = Sy_2$, we obtain vectors with the properties stated. (iii) is proved in a similar manner as (ii). (iv) If $A_1x_1 = A_2x_2$, then $x_1 = A_1^{-1}A_2x_2$ and since $A_1^{-1}A_2$ is a P-matrix, the result follows from the characterization by Fiedler and Pták [2]. (v) follows in a similar way from the fact that $A_1A_2^{-1}$ is a P-matrix. ∎

The necessary and sufficient nonsingularity conditions given in Theorem 2 are generally very difficult to verify. This fact becomes more understandable in the light of the recent result by Poljak and Rohn [4] stating that testing nonsingularity of an interval matrix is an NP-complete problem.

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Souhrn

REGULARITA A *P*-MATICE

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Jsou uvedeny nové důkazy dvou dříve publikovaných vět o vztahu regularity intervalových matic k reálným *P*-maticím.

Резюме

РЕГУЛЯРНОСТЬ И *P*-МАТРИЦЫ

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В статье приведены новые доказательства двух ранее опубликованных теорем о взаимоотношении регулярных интервальных матриц и *P*-матриц.

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