Nonsingularity Under Data Rounding

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ABSTRACT

For a square floating point matrix A we give a test for the existence of a singular real matrix which rounds to A.

In this paper, which may be considered a footnote to [2], we show how a result concerning interval matrices can be applied to handle, at least theoretically, the problem of nonsingularity of real matrices in a floating point representation. We give the results for the decimal floating point system; however, the technique used applies easily to other systems (e.g. binary or hexadecimal) as well.

Let $A = (a_{ij})$ be an $n \times n$ floating point matrix in a decimal floating point system with normalized mantissa length d decimal digits. The matrix A represents in fact a whole class $C(A)$ of real matrices which round to A. Obviously, A can be considered numerically nonsingular if all matrices in $C(A)$ are nonsingular, and numerically singular if $C(A)$ contains a singular matrix, since in the latter case A cannot be distinguished from a singular matrix by means of our floating point system.

It follows from the properties of correct rounding that the class $C(A)$ can be described as

$$C(A) = \{A';\ A - 10^{-d}A_2 \leqslant A' < A + 10^{-d}A_1\},$$

where, with the coefficients of A written in the form

$$a_{ij} = \pm\, m_{ij} 10^{p_{ij}} \qquad (i, j = 1, \ldots, n),$$

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the matrices $A_1 = (a^1_{ij})$, $A_2 = (a^2_{ij})$ are given by

$$a^1_{ij} = 0.5 \times 10^{p_{ij}} \qquad (i, j = 1, \dots, n)$$

$$a^2_{ij} = 0.5 \times 10^{p_{ij}} \qquad \text{if} \quad m_{ij} \in (0.1, 1)$$

$$a^2_{ij} = 0.5 \times 10^{p_{ij}-1} \quad \text{if} \quad m_{ij} = 0.1.$$

Let us denote

$$A_c = A + \tfrac{1}{2} 10^{-d}(A_1 - A_2),$$

$$A_r = \tfrac{1}{2}(A_1 + A_2),$$

so that $A_r \geqslant 0$. Then $C(A)$ can be written in a more appropriate centered form

$$C(A) = \{A';\ A_c - 10^{-d}A_r \leqslant A' < A_c + 10^{-d}A_r\}.$$

Let us now introduce, assuming that A_c is nonsingular, the number

$$c(A) = \log_{10} \max\{\rho_0(A_c^{-1}S_1A_rS_2);\ S_1, S_2 \text{ signature matrices}\}, \qquad (1)$$

where a *signature matrix* is a diagonal matrix all of whose diagonal entries are equal to $+1$ or -1, and for a square matrix B, $\rho_0(B)$ is defined by

$$\rho_0(B) = \max\{|\lambda|;\ Bx = \lambda x,\ x \neq 0,\ \lambda \text{ real}\}.$$

Hence ρ_0 is an analogue of the spectral radius of B, with the maximum being taken only over real eigenvalues; we define $\rho_0(B) = 0$ if no real eigenvalue exists.

We shall show in the next theorem that $c(A)$ is correctly defined and is closely connected to our problem of testing numerical (non)singularity:

THEOREM. *Let A be a floating point matrix such that A_c is nonsingular. Then $c(A)$ is well defined and we have:*

(i) *if $c(A) < d$, then each matrix in $C(A)$ is nonsingular;*
(ii) *if $c(A) > d$, then $C(A)$ contains a singular matrix.*

COMMENTS. *The case of $c(A)=d$ remains undecided, but it can scarcely occur, since $c(A)$ is generally a real number, while d is integer. Also, notice that if no coefficient of A is of the form 10^p, p integer, then $A_c = A$, so that in this case the assumption of the theorem reduces to nonsingularity of A.*

Proof. For a positive real number β, consider the interval matrix

$$M_\beta = \{A';\ A_c - \beta A_r \leqslant A' \leqslant A_c + \beta A_r\}.$$

According to assertion (C3) of Theorem 5.1 in [2], M_β contains a singular matrix if and only if

$$\beta \rho_0\left(A_c^{-1} S_1 A_r S_2\right) \geqslant 1$$

holds for some signature matrices S_1, S_2. Since $|A_c| \leqslant 2A_r$ (as can be easily verified), it follows that M_2 contains the zero matrix; thus $\rho_0(A_c^{-1}S_1A_rS_2) \geqslant \frac{1}{2}$ for some signature matrices S_1, S_2, which shows that the maximum in (1) is positive; hence $c(A)$ is *well defined.* Next we prove (i) and (ii)

(i) If $c(A) < d$, then $10^{-d}\rho_0(A_c^{-1}S_1A_rS_2) < 1$ for any signature matrices S_1, S_2, which in view of the above-quoted theorem shows that $M_{10^{-d}}$ consists only of nonsingular matrices. Since $C(A) \subset M_{10^{-d}}$, the assertion follows.

(ii) If $c(A) > d$, then $10^{-d}\rho_0(A_c^{-1}S_1A_rS_2) > 1$ holds for some signature matrices S_1, S_2. Take a $\beta \in (0, 10^{-d})$ such that $\beta\rho_0(A_c^{-1}S_1A_rS_2) > 1$ still holds. Then M_β contains a singular matrix, and thus so does $C(A)$, since $M_\beta \subset C(A)$ in view of $\beta < 10^{-d}$. ■

The result, although solving the problem in principle, remains merely of theoretical interest due to the complicated form of the formula (1) (note that there are 2^n signature matrices of size n). Nevertheless, in the special case of nonnegative invertible matrices the formula for $c(A)$ can be given a very simple form (ρ denotes the spectral radius):

COROLLARY. *Let $A_c^{-1} \geqslant 0$. Then we have*

$$c(A) = \log_{10} \rho\left(A_c^{-1} A_r\right). \tag{2}$$

Proof. With E the unit matrix, from the Perron-Frobenius theorem

[1] we obtain that

$$\begin{aligned} c(A) &= \log_{10} \max_{S_1,S_2} \rho_0\left(A_c^{-1}S_1A_rS_2\right) \\ &\leqslant \log_{10} \max_{S_1,S_2} \rho\left(|A_c^{-1}S_1A_rS_2|\right) \leqslant \log_{10} \rho\left(A_c^{-1}A_r\right) \\ &= \log_{10} \rho_0\left(A_c^{-1}EA_rE\right) \leqslant c(A), \end{aligned}$$

which proves (2). ■

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REFERENCES

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