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A Theorem on *P*-Matrices

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We give a new proof of a theorem stating that given two square matrices such that their interval hull contains no singular matrix, then each of them can be obtained from the other one by premultiplying it (from left or right) by a P-matrix.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices. The interval hull of A and B, denoted by $\text{Int}(A, B)$, is defined as the set of matrices $C = (c_{ij})$ satisfying

$$\min\{a_{ij}, b_{ij}\} \le c_{ij} \le \max\{a_{ij}, b_{ij}\}$$

for $i, j = 1, \ldots, n$. We are concerned here with a proof of the following result:

THEOREM *Let A and B be two $n \times n$ matrices such that* $\text{Int}(A, B)$ *contains no singular matrix. Then there exist P-matrices Q_1, R_1, Q_2, R_2 such that*

$$B = Q_1 A = A R_1$$
$$A = Q_2 B = B R_2$$

hold.

Before proceeding to the proof, we shall give some comments. First, recall that a square matrix is said to be a P-matrix if all its principal minors are positive [2]. Second, the value of this theorem consists in the fact that it links the linear complementarity theory to the problem of solving systems of linear equations with inexact data. In fact, it was shown in [6] that each vertex of the convex hull of the solution set of a system of linear equations with interval-valued data (which is generally nonconvex) is a solution of a problem of the form

$$Ay = Bx + b \tag{1.1}$$

$$y \ge 0, \qquad x \ge 0 \tag{1.2}$$

$$y^T x = 0 \tag{1.3}$$

where A, B belong to an interval matrix $A^I = \{A';\ \underline{A} \le A' \le \overline{A}\}$ describing the uncertainties of the coefficients of the linear system. Hence the Theorem, under the assumption of nonsingularity of each $A' \in A^I$, gives that $A^{-1}B$ is a P-matrix, which in conjunction with the well-known result by Samelson, Thrall and Wesler [8] assures that (1.1)–(1.3) has a unique solution. This, on one hand, shows that the vertices can be found by solving the linear complementarity problem

$$y = A^{-1}Bx + A^{-1}b$$
$$y \geq 0, \qquad x \geq 0$$
$$y^T x = 0$$

by standard algorithms (Cottle and Dantzig [1], Lemke [4], Murty [5]) which are all known to solve finitely linear complementarity problems with P-matrices. On the other hand, it was shown in [6] that a number of necessary and sufficient nonsingularity conditions for interval matrices can be formulated in terms of solutions of problems of type (1.1)–(1.3).

The Theorem was first proved in another formulation in [6, Theorem 1.2] via an auxiliary result on interval matrices. It was reproved (in the same formulation) in [7] in another way, using no properties of P-matrices but the definition. We are going to give here a new proof which, in our view, is the most direct one of all three of them.

Proof of the Theorem For the purposes of the proof, let C_i denote the ith row of a matrix C. It follows from the assumption that both A and B are nonsingular. We shall first prove that $Q_1 = BA^{-1}$ is a P-matrix. Assume to the contrary it is not the case; then, as proved by Fiedler and Pták [2] or Gale and Nikaido [3], there exists a vector x, $x \neq 0$, such that $x_i(BA^{-1}x)_i \leq 0$ for each i. Define a matrix C by rows for $i = 1, \ldots, n$ by

$$C_i = B_i + t_i(A_i - B_i) \tag{2}$$

where $t_i = 1$ if $x_i = 0$ and if $x_i \neq 0$, then t_i is an arbitrary root of the continuous function of one real variable

$$\varphi_i(t) = x_i(B + t(A - B))_i A^{-1} x$$

in $[0, 1]$; such a root exists since $\varphi_i(0) = x_i(BA^{-1}x)_i \leq 0$ and $\varphi_i(1) = x_i^2 \geq 0$. Then it follows from (2) that C_i is a convex combination of A_i and B_i for each $i \in \{1, \ldots, n\}$; hence $C \in \text{Int}(A, B)$. Next we prove that

$$CA^{-1}x = 0 \tag{3}$$

holds; in fact, if $x_i = 0$, then $(CA^{-1}x)_i = A_i A^{-1}x = x_i = 0$, and if $x_i \neq 0$, then $(CA^{-1}x)_i = C_i A^{-1}x = (1/x_i)\varphi_i(t_i) = 0$. Therefore (3) implies that C is singular; this contradiction shows that Q_1 is a P-matrix. Now, since $\text{Int}(A^T, B^T) = \{C^T;\ C \in \text{Int}(A, B)\}$ does not contain a singular matrix, from the result just proved we obtain that $B^T = Q_1' A^T$ for some P-matrix Q_1', hence $B = AR_1$ where $R_1 = (Q_1')^T$ is a P-matrix.

The second set of equations follows from the first one applied to $\text{Int}(B, A)$ which is equal to $\text{Int}(A, B)$. ∎

We have this corollary:

COROLLARY *Let A, B be two nonsingular matrices such that at least one of the matrices $A^{-1}B, AB^{-1}, B^{-1}A, BA^{-1}$ is not a P-matrix. Then* $\mathrm{Int}(A, B)$ *contains a singular matrix.*

Proof Follows immediately from the Theorem. ∎

Some other consequences of the Theorem are listed in [7].

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