ON THE COMMON ARGUMENT BEHIND THE FINITE PIVOTING RULES BY BLAND AND MURTY

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We show that the finiteness of both Bland's pivoting rule for linear programming and Murty's pivoting rule for the linear complementarity problem can be established from an auxiliary result stating finiteness of certain sequences of positive integers.

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Murty [2] formulated in 1974 a finite pivoting rule for solving a linear complementarity problem

$$\begin{aligned} y &= Mx + q \\ y^T x &= 0 \\ y &\geqslant 0,\ x \geqslant 0 \end{aligned} \qquad (1)$$

with a P-matrix M. Three years later, Bland [1] published his finite pivoting rule for solving a linear programming problem of size $m \times n$

$$\min \left\{ c^T x;\ Ax = b,\ x \geqslant 0 \right\} . \qquad (2)$$

Both the rules can be given a verbally identical formulation as follows:

The rule. Among all variables eligible for entering (leaving) the basis choose that one having the minimum index.

(In Murty's algorithm the leaving variable is determined automatically, hence the rule applies to entering variable only). It is the purpose of this note to demonstrate that the finiteness of both the rules follows from this general result:

Lemma. Let S be a sequence having the following two properties:

(i) each member of S belongs to $\{1,\dots,n\}$, $n \geq 1$,

(ii) between each two appearances of any j in S there is an appearance of some k, $k > j$, in S.

Then the sequence S is finite, having at most $2^n - 1$ members, and each $j \in \{1,\dots,n\}$ appears there at most 2^{n-j} times.

Proof. We prove by induction on $j = n, n-1, \dots, 1$ that j can appear at most 2^{n-j} times in S. The case of $j = n$ is obvious since the assumption (ii) precludes n to appear more than once in S. So let $j < n$. In view of (ii), the number of appearances of j in S cannot be greater than the number of appearances of all the numbers greater than j in S, increased by 1; due to the inductive assumption, the latter value is bounded from above by the number $\sum_{k=j+1}^{n} 2^{n-k} + 1 = 2^{n-j}$, which concludes the inductive part of the proof. Hence S is finite, consisting of at most $\sum_{j=1}^{n} 2^{n-j} = 2^n - 1$ members. ∎

The finiteness of Murty's pivoting rule for solving (1) was proved by this author in [3] using the Lemma (although not formulated explicitly there) applied to the set S of indices of variables entering the basis in the course of Murty's algorithm. We are going to show here that it is also the case of Bland's rule for solving (2) if S is chosen as the set of indices of entering variables during a stall (period when the objective is kept constant) in the simplex algorithm. To this end, we need the following auxiliary result whose proof is a variant of the original Bland's idea in [1] (to facilitate formulations, we identify the basis with the set of indices of basic variables):

Proposition. Assume that some j enters and later leaves the basis in the course of the simplex algorithm using Bland's pivoting rule. Then either the objective decreases, or some k, $k > j$, enters the basis between the two steps.

Proof. Assume the algorithm stalls between the two steps. Let B_o and B be the bases the index j enters and leaves, respectively, and let s enters B instead of j. Denote, as customary, $\bar{A} = A_B^{-1}A$, $\bar{b} = A_B^{-1}b$, $\bar{c} = c^T - c_B^T\bar{A}$. Let y be the criterial row corresponding to the basis B_o and let z (of the size of y) be defined as follows: $z_B = A_{\cdot s}$ (the s-th column of $\bar{A}$), $z_s = -1$, $z_j = 0$ otherwise. Then $Az = 0$, so that $yz = c^Tz = -\bar{c}_s > 0$, which gives that there is a k with $y_k z_k > 0$. Hence $k \notin B_o$ and since $y_j < 0$ and $z_j > 0$, it must be $k \neq j$. We prove that $k > j$; two cases may occur:

(a) If $y_k < 0$ and $z_k < 0$, then k was eligible for entering B_o, but not chosen, which means that $k > j$. Since $z_k < 0$, it gives that either k is in the basis, or $k = s$, hence just entering it. In both the cases, k is in the basis when j is pivoted out of it.

(b) Let $y_k > 0$ and $z_k > 0$. Then $k \in B$, let $k = B_p$. Since k entered the basis during a stall, we have $\bar{b}_p = 0$ and $\bar{a}_{ps} = z_k > 0$, hence k was eligible for leaving the basis, but not chosen, which again implies $k > j$. ∎

Now we can establish the finiteness of Bland's rule:

Theorem. The simplex method using Bland's pivoting rule is finite.

Proof. Assume some basis B reappears in the course of the algorithm. Denote by S the sequence of indices entering the basis from the first appearance of B on. Then S is infinite since the algorithm cycles;

but S is also finite since it satisfies the assumptions of the Lemma in view of the Proposition (if some j appears in S twice, then it must have left the basis in the meantime and therefore some $k > j$ meanwhile entered the basis). This contradiction shows that no basis can reappear, hence the algorithm is finite since the number of bases is finite. ∎

As a by-product we have also proved that no j can enter the basis more than 2^{n-j} times during a stall in the algorithm, a fact mentioned by Bland in [1, p. 104].

References

[1] R. G. Bland, ''New finite pivoting rules for the simplex method'', Math. of Oper. Res. 2(1977) 103-107.

[2] K. G. Murty, ''Note on a Bard-type scheme for solving the complementarity problem'', Opsearch 11(1974) 123-130.

[3] J. Rohn, ''A short proof of finiteness of Murty's principal pivoting algorithm'', Math. Program. 46(1990) 255-256.