where

$$\Theta_i = \frac{1}{\sqrt{1-|k_i|^2}}\begin{pmatrix} 1 & -k_i \\ -\bar{k}_i & 1 \end{pmatrix}.$$

Comparing (10) and (A.2) we can see that the ith column of the factor Y in (15) can be easily calculated from the array in the form

$$y_i = \sqrt{1-|\xi_i|^2}\,\delta_i.$$

A major simplification occurs when $\xi_i = 0$. In this case, the second step of the recursion (A.2b) disappears and the recursion consists of only two steps: 1) a multiplication of the array from the left by the matrix Θ_i and ii) a multiplication from the right of one of the columns by the matrix T^*. A similar result holds for the case $\epsilon_i = -1$.

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Stability of Interval Matrices: The Real Eigenvalue Case

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Abstract—**Hollot and Bartlett showed that testing at most 2^{n^2} certain matrices for stability is sufficient for verifying stability of an $n \times n$ interval matrix with real eigenvalues. We prove that this upper bound can be reduced to 2^{2n-1} and consider a special case where testing only two matrices is needed.**

I. INTRODUCTION

Let $\underline{A}$ and $\bar{A}$ be real $n \times n$ matrices, $\underline{A} \le \bar{A}$. The set of matrices $[\underline{A}, \bar{A}] = \{A;\ \underline{A} \le A \le \bar{A}\}$, termed an interval matrix, is said to be stable if each $A \in [\underline{A}, \bar{A}]$ is stable, i.e., has all its eigenvalues in the open left-half of the complex plane.

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The problem of stability of interval matrices has been recently extensively studied due to its applications in control theory, e.g., cf. [2] and the references contained therein. In this note, we shall be concerned with a special case of interval matrices $[\underline{A}, \bar{A}]$ such that each $A \in [\underline{A}, \bar{A}]$ has only real eigenvalues. Under this assumption we shall delineate a finite family of matrices in $[\underline{A}, \bar{A}]$ such that this interval matrix is stable if and only if each matrix in the family is stable. This result is based on some characterization of singular interval matrices given in [3].

Let us introduce the set

$$L = \{\lambda \in R^1;\ \det(A - \lambda I) = 0 \text{ for some } A \in [\underline{A}, \bar{A}]\}$$

where det, as usual, denotes the determinant and I is the unit matrix. Further, let

$$\bar{\lambda} = \max_L \lambda.$$

Then, under our assumption that all the eigenvalues are real, we have that $[\underline{A}, \bar{A}]$ is stable if and only if $\bar{\lambda} < 0$; this, of course, is still a mere reformulation of the definition.

Hollot and Bartlett proved in [1, corollary 2] that $\bar{\lambda}$ is an eigenvalue of some matrix A of the form

$$A_{ij} \in \{\underline{A}_{ij}, \bar{A}_{ij}\} \qquad (i, j = 1,\cdots,n) \tag{1}$$

hence, the stability of $[\underline{A}, \bar{A}]$ with real eigenvalues is equivalent to the stability of all matrices of type (1). In the worst (but quite possible) case of $\underline{A} < \bar{A}$ (componentwise) there are altogether 2^{n^2} mutually different matrices of the form (1). We shall show that this upper bound can be essentially reduced (remaining, nevertheless, an exponential one) by considering a certain subset of matrices of type (1). To this end, denote

$$Y = \{y \in R^n;\ y_j = \pm 1 \text{ for } j = 1,\cdots,n\}$$

and for each $y, z \in Y$ define a matrix A_{yz} by

$$(A_{yz})_{ij} = \underline{A}_{ij} \qquad \text{if } y_i z_j = 1$$

$$(A_{yz})_{ij} = \bar{A}_{ij} \qquad \text{if } y_i z_j = -1$$

$(i, j = 1,\cdots,n)$. Obviously, each A_{yz} is of the form (1) and belongs to $[\underline{A}, \bar{A}]$. It can be easily seen that in the worst case $\underline{A} < \bar{A}$ there are exactly 2^{2n-1} mutually different matrices A_{yz} since the cardinality of Y is 2^n and $A_{-y,-z} = A_{yz}$ for each $y, z \in Y$. First, we have this result.

Theorem 1: Let $\lambda \in R^1$. Then $\lambda \in L$ if and only if

$$\det(A_{yz} - \lambda I)\det(A_{y'z'} - \lambda I) \le 0 \tag{2}$$

holds for some $y, z, y', z' \in Y$.

Proof: $\lambda \in L$ if and only if $\det(A - \lambda I) = 0$ for some $A \in [\underline{A}, \bar{A}]$ which is the case if and only if the interval matrix $[\underline{A} - \lambda I, \bar{A} - \lambda I]$ contains a singular matrix; the latter fact is equivalent to (2) in view of the assertion (C1) of [3, theorem 5.1]. □

Now we can prove the main result.

Theorem 2: Let $[\underline{A}, \bar{A}]$ have only real eigenvalues. Then $[\underline{A}, \bar{A}]$ is stable if and only if each matrix $A_{yz}, y, z \in Y$, is stable.

Proof: The "only if" part is obvious. To prove the "if" part, first notice that $\bar{\lambda} \in L$, hence, it satisfies

$$\det\left(A_{yz} - \bar{\lambda} I\right)\det\left(A_{y'z'} - \bar{\lambda} I\right) \le 0 \tag{3}$$

for some $y, z, y', z' \in Y$ according to Theorem 1. Assume to the contrary that the left-hand side in (3) is strictly negative. Then in view of the continuity of the determinant there exists a positive ϵ

such that each $\lambda \in (\bar{\lambda} - \epsilon, \bar{\lambda} + \epsilon)$ satisfies (2) and therefore belongs to L, which gives that $\bar{\lambda}$ is an interior point of L contrary to its definition as $\bar{\lambda} = \max_L \lambda$. Hence, the left-hand side in (3) is zero, which means that $\bar{\lambda}$ is an eigenvalue of either A_{yz} or $A_{y'z'}$, in both the cases $\bar{\lambda} < 0$ in view of the stability of these matrices. Hence, $[\underline{A}, \bar{A}]$ is stable. □

In this way, we have reduced the upper bound on the number of matrices to be tested for stability from 2^{n^2} to 2^{2n-1}. This, of course, is still exponential in n. Nevertheless, we shall delineate a class of interval matrices for which only *two* matrices of type A_{yz} are to be tested for stability. This class is specified by the following four properties.

A1) Each $A \in [\underline{A}, \bar{A}]$ has n real eigenvalues numbered in such a way that $\lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_n(A)$.

A2) $\lambda_{n-1}(A') < \lambda_n(A)$ for each $A', A \in [\underline{A}, \bar{A}]$.

A3) There are $y, z \in Y$ such that for each $A \in [\underline{A}, \bar{A}]$ there exists an eigenvector x and a left eigenvector p, both pertaining to $\lambda_n(A)$, with $z_i x_i > 0$, $y_i p_i > 0$ for $i = 1, \cdots, n$.

A4) $\bar{A} - \underline{A}$ is a positive matrix of rank one.

In view of A2), each $\lambda_n(A)$ is simple and, therefore, both its eigenspace and its left eigenspace are one-dimensional. Therefore, A3) states that the sign pattern of any eigenvector pertaining to an nth eigenvalue is described by z or $-z$; similarly by y or $-y$ for the left eigenvector.

Theorem 3: Let $[\underline{A}, \bar{A}]$ be an interval matrix satisfying A1), A2), A3), A4). Then $[\underline{A}, \bar{A}]$ is stable if and only if both the matrices A_{yz} and $A_{-y,z}$ are stable.

Comment: In contrast to Theorem 2, here y and z are fixed vectors prescribed in the assumption A3).

Proof: Under the assumptions stated it was shown in [4, proof of theorem 3] that $\bar{\lambda} = \max\{\lambda_n(A);\ A \in [\underline{A}, \bar{A}]\} = \max\{\lambda_n(A_{yz}), \lambda_n(A_{-y,z})\}$. Hence, if A_{yz} and $A_{-y,z}$ are stable, then $\lambda_n(A_{yz}) < 0$, $\lambda_n(A_{-y,z}) < 0$ and thus $\bar{\lambda} < 0$, so that $[\underline{A}, \bar{A}]$ is stable. The converse statement is obvious. □

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Global Tunability of One-Dimensional SISO Systems

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***Abstract*—It is shown by example that with a suitably defined certainty equivalence controller Σ_R, it is possible to make a closed-loop parameterized system Σ tunable on its entire parameter space $\mathscr{P}$, even though $\mathscr{P}$ may contain points at which the parameterized model Σ_D upon which Σ_R's definition is based, is not stabilizable. The implications of this discovery are briefly discussed.**

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I. INTRODUCTION

One way to think of a parameter adaptive control system is as the feedback interconnection of a process Σ_P and a parameterized controller $\Sigma_C(k)$ whose parameter vector k is adjusted by a tuner Σ_T [1]. From this point of view, a parameterized controller is a dynamical system depending on k, whose inputs are the process open-loop control u, the process output y, and possibly a reference input r; Σ_C's outputs invariably include not only a control signal u_C which in closed-loop serves as the feedback control u to the process, but also a suitably defined "tuning error" e_T which during adaptation drives Σ_T. Σ_C typically consists of two subsystems, one an "identifier" $\Sigma_I(k)$ whose primary function is to generate an "identification error" e_I and the other an "internal regulator" $\Sigma_R(k)$ whose output is u_C; in this case e_T is usually the same as e_I. Σ_R is often chosen in accordance with the certainty equivalence principle: this is done by defining Σ_R's coefficient matrices as functions of k so that at each point p in the parameter space $\mathscr{P}$ in which k takes values, the closed-loop system $\Sigma_{D_{cl}}(p)$ consisting of $\Sigma_R(p)$ in feedback with Σ_I's design model $\Sigma_D(p)$ is internally stable (cf., [1]). As is well known, the difficulty with this approach is that $\mathscr{P}$ usually contains points at which Σ_D cannot be stabilized; this of course means that an internally stabilizing certainty equivalence control of this type cannot be defined on any subset of $\mathscr{P}$ containing such points. The purpose of this note is to expand on a recently proposed idea [3], which has promise for dealing with this problem.

To explain what we have in mind, we shall appeal to the concept of tunability, a property of the closed-loop system $\Sigma(k)$ consisting of Σ_P in feedback with $\Sigma_C(k)$. Σ is said to be *tunable* on a nonempty subset $\mathscr{E} \subset \mathscr{P}$ if for each $p \in \mathscr{E}$, Σ's state x goes to zero as $t \to \infty$ along each trajectory on which $k(t) = p$ and e_T together with any exogenous inputs to Σ (e.g., r) equal zero. It can be shown quite easily for the case in which $\Sigma(k)$ is a linear system whose coefficient matrices are "locally bounded" functions[1] of k, that if Σ's exogenous inputs are bounded, if Σ is tunable on $\mathscr{P}$, and if e_T can be kept within finite bounds by slowly tuning k, then Σ's state x will remain bounded as well. The reason why this is important here, is that Σ_I and Σ_R can often be designed so that Σ has these properties. In particular, it is possible to define Σ_R so that Σ is tunable on $\mathscr{P}$, even though $\mathscr{P}$ may contain points at which Σ_D is not stabilizable! This will be demonstrated in the sequel for the case when Σ_P is a one-dimensional, SISO, linear system.

II. FORMULATION

The problem of interest is to construct an adaptive stabilizer for a SISO process Σ_P with input u and output y, which is modeled by a one-dimensional linear system of the form

$$\dot{y} = ay + bu \tag{1}$$

where a and b are unknown constants. We assume throughout that $|b|$ is bounded below by a known positive number b^*; that is

$$|b| \ge b^*. \tag{2}$$

[1] A function is *locally bounded* if it is bounded on each bounded subset of its domain.