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INTERVAL MATRICES: SINGULARITY AND REAL EIGENVALUES*

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Abstract. This paper proves that a singular interval matrix contains a singular matrix of a very special form. This result is applied to study the real part L of the spectrum of an interval matrix. Under the assumption of sign stability of eigenvectors this paper gives a complete description of L by means of spectra of a finite subset of matrices and formulates a stability criterion for interval matrices with real eigenvalues that requires checking only two matrices for stability.

Key words. interval matrix, singular matrix, eigenvalue, stability

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1. Introduction. Let A_c and Δ be real $n \times n$ matrices with Δ nonnegative. The set of matrices

$$[A_c - \Delta, A_c + \Delta] := \{A; A_c - \Delta \leqq A \leqq A_c + \Delta\}$$

is called an interval matrix and is said to be singular if it contains a singular matrix. The problem of singularity of interval matrices is studied in [6], where in Theorem 5.1 a number of necessary and sufficient singularity conditions are given.

The purpose of the present paper is two-fold. First, we prove in Theorem 2.2 that a singular interval matrix $[A_c - \Delta, A_c + \Delta]$ contains a singular matrix A of a very special form

$$A = A_c - dT_y \, \Delta T_z, \tag{0}$$

where $d \in [0, 1]$ and T_z, T_y are diagonal matrices whose vectors of diagonal entries are the sign vectors of some singular vectors x and p of A and A^t, respectively. Second, in § 3 we use this theorem to study the properties of the set L of *real* eigenvalues of all matrices contained in a given interval matrix. We prove that each $\lambda \in L$ is an eigenvalue of some matrix of the form (0) (Theorem 3.2). Moreover, if $\lambda \in \partial L$, then $d = 1$ (Theorem 3.4); hence each boundary point of L is achieved at some vertex of $[A_c - \Delta, A_c + \Delta]$ (considered a polyhedron in R^{n^2}). To obtain more specific results, we introduce three assumptions imposing sign stability restrictions on eigenvectors under which we give in Theorem 3.7 a complete description of the set L as a union of at most n compact intervals whose endpoints are eigenvalues of some explicitly expressed matrices. All the results are formulated for the real part of the spectrum only, since the complex case seemingly cannot be handled by the methods used.

Stability of interval matrices has been recently extensively studied in robust control theory; see the state-of-the-art papers by Mansour [4] and Barmish [1] for detailed information. The description of the set L in Theorem 3.7 here implies a simple stability criterion in the case when only real eigenvalues are present: under the three assumptions made, an interval matrix is stable if and only if two explicitly given matrices are stable (Theorem 3.8).

We shall use the following notation. The absolute-value vector of a vector $x = (x_i)$ is defined by $|x| = (|x_i|)$. We introduce the set

$$Y = \{y \in R^n; y_j = \pm 1 \text{ for } j = 1, \ldots, n\},$$

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and for each $y \in Y$ we denote by T_y the $n \times n$ diagonal matrix with diagonal vector y. Inequalities, such as $\Delta \geqq 0$ or $\Delta > 0$, are to be understood componentwise. A^t denotes the transpose of A.

2. Singular interval matrices. Theorem 2.2 below establishes the main result of this paper; all subsequent theorems are consequences of it. It will be preceded by an auxiliary characterization of singular interval matrices.

LEMMA 2.1. *A square interval matrix* $[A_c - \Delta, A_c + \Delta]$ *is singular if and only if it satisfies*

$$|A_c x| \leqq \Delta |x|$$

for some nonzero vector x.

Proof. The assertion is an immediate consequence of the theorem by Oettli and Prager [5] that characterizes solutions of systems of linear interval equations (in our case, with zero right-hand sides); we use it here in the form given, e.g., in [6, Thm. 2.1]. □

Now we have the following main result.

THEOREM 2.2. *Let* $[A_c - \Delta, A_c + \Delta]$ *be a singular interval matrix. Then there exist* $x \neq 0$, $p \neq 0$, *and* $y, z \in Y$ *such that*

$$(A_c - dT_y \Delta T_z)x = 0, \tag{1}$$

$$(A_c - dT_y \Delta T_z)^t p = 0, \tag{2}$$

$$T_z x \geqq 0, \tag{3}$$

and

$$T_y p \geqq 0 \tag{4}$$

hold, where

$$d = \min \{\varepsilon \geqq 0;\ [A_c - \varepsilon\Delta, A_c + \varepsilon\Delta] \text{ is singular}\} \tag{5}$$

and $d \in [0, 1]$.

Comment. Notice that (3) and (4) mean that $z_j x_j \geqq 0$, $p_j y_j \geqq 0$ hold for $j = 1, \ldots, n$. Hence if all entries of x and p are nonzero, then z and y are uniquely determined and their entries are simply the signs of the respective entries of x and p. Also notice that (3) and (4) imply $T_z x = |x|$ and $T_y p = |p|$.

Proof. First observe that (1)–(4) hold trivially if A_c is singular. In this case $d = 0$; hence it suffices to take some nonzero vectors x, p that satisfy $A_c x = 0$, $A_c^t p = 0$ and choose z and y as the sign vectors of x and p; then (1)–(4) hold.

Let A_c be nonsingular, so that $d > 0$. Since the interval matrix $[A_c - d\Delta, A_c + d\Delta]$ is singular, according to the assertion (C1) of [6, Thm. 5.1], there exist $y, z \in Y$ such that $\det(A_c - dT_y \Delta T_z) \cdot \det A_c \leqq 0$. Then the continuous function φ of one real variable $\varphi(\tau) = \det(A_c - \tau d T_y \Delta T_z)$ satisfies $\varphi(0)\varphi(1) \leqq 0$; hence $\varphi(\tau_0) = 0$ for some $\tau_0 \in [0, 1]$. In view of (5) it must be $\tau_0 d \geqq d$; hence $\tau_0 = 1$, so that $A_c - dT_y \Delta T_z$ is singular and (1) holds for some $x \neq 0$. To prove the assertions (2)–(4), we shall distinguish two cases: (a) $\Delta > 0$ and (b) $\Delta \geqq 0$.

(a) Let $\Delta > 0$.

(i) To prove (3), assume to the contrary that neither $T_z x \geqq 0$ nor $T_z x \leqq 0$ holds, so that there exist $j, k \in \{1, \ldots, n\}$ such that $z_j x_j < 0$ and $z_k x_k > 0$. Take an arbitrary

$i \in \{1, \ldots, n\}$. Then one of the numbers $y_i z_j x_j$, $y_i z_k x_k$ is positive and the other is negative, which gives

$$|(A_c x)_i| = d\left|\sum_h \Delta_{ih} y_i z_h x_h\right| < d(\Delta|x|)_i;$$

since i was arbitrary, it follows that $|A_c x| < d\Delta|x|$. Then we can choose a positive ε such that $\varepsilon < d$ and $|A_c x| < \varepsilon\Delta|x|$; hence the interval matrix $[A_c - \varepsilon\Delta, A_c + \varepsilon\Delta]$ is singular according to Lemma 2.1, which contradicts (5). This contradiction shows that either $T_z x \geqq 0$ or $T_z x \leqq 0$ holds. In the former case we are done, and in the latter it is sufficient to set $x := -x$ to obtain (1) and (3).

(ii) To prove (2) and (4), we first notice that since $A_c - dT_y\Delta T_z$ is singular, there exists a nonzero vector p such that

$$(A_c - dT_y\Delta T_z)^t p = (A_c^t - dT_z\Delta^t T_y)p = 0.$$

Assume that neither $T_y p \geqq 0$ nor $T_y p \leqq 0$ holds. Arguing as in (i), we obtain that $|A_c^t p| < d\Delta^t|p|$, which again gives that the interval matrix $[A_c^t - \varepsilon\Delta^t, A_c^t + \varepsilon\Delta^t]$ is singular for some $\varepsilon < d$; hence so is $[A_c - \varepsilon\Delta, A_c + \varepsilon\Delta]$, which is a contradiction. Hence either $T_y p \geqq 0$ or $T_y p \leqq 0$, so that by setting $p := -p$ if necessary we get that (2) and (4) hold also. This concludes the proof for the case of $\Delta > 0$.

(b) Let Δ be a nonnegative matrix. Let H denote the matrix of all ones and for $k = 1, 2, \ldots$ define $\Delta_k = \Delta + (1/k)H$; then $\Delta_k > 0$ and each $[A_c - \Delta_k, A_c + \Delta_k]$ is singular. Hence from what has been proved under (a), it follows that for each k there exist vectors x_k, p_k (which can be normalized so that $\|x_k\|_2 = \|p_k\|_2 = 1$), and $z_k, y_k \in Y$ such that

(1′) $$(A_c - d_k T_{y_k}\Delta_k T_{z_k})x_k = 0,$$

(2′) $$(A_c - d_k T_{y_k}\Delta_k T_{z_k})^t p_k = 0,$$

(3′) $$T_{z_k} x_k \geqq 0,$$

(4′) $$T_{y_k} p_k \geqq 0,$$

where

(5′) $$d_k = \min\{\varepsilon \geqq 0;\ [A_c - \varepsilon\Delta_k, A_c + \varepsilon\Delta_k] \text{ is singular}\}.$$

First we show that $d_k \leqq d_{k+1} \leqq d$ for $k = 1, 2, \ldots$. In fact, from $\Delta_k \geqq \Delta_{k+1} \geqq \Delta$ it follows that $[A_c - d_{k+1}\Delta_{k+1}, A_c + d_{k+1}\Delta_{k+1}] \subset [A_c - d_{k+1}\Delta_k, A_c + d_{k+1}\Delta_k]$, and $[A_c - d\Delta, A_c + d\Delta] \subset [A_c - d\Delta_{k+1}, A_c + d\Delta_{k+1}]$, which implies that both the interval matrices $[A_c - d_{k+1}\Delta_k, A_c + d_{k+1}\Delta_k]$ and $[A_c - d\Delta_{k+1}, A_c + d\Delta_{k+1}]$ are singular; hence $d_k \leqq d_{k+1}$ and $d_{k+1} \leqq d$ in view of (5′). Next, since Y is finite, there exists a constant subsequence of the sequence $\{(z_k, y_k)\}_{k=1}^{\infty}$, i.e., $z_k = z$, $y_k = y$ for infinitely many k. Let us choose another subsequence of this subsequence along which x_k, p_k converge to some x, p (this is possible since $\{x_k\}$, $\{p_k\}$ are confined to the compact unit sphere; hence $x \neq 0$ and $p \neq 0$). Then taking limits in (1′)–(4′) we obtain

$$(A_c - \bar{d}T_y\Delta T_z)x = 0,$$

$$(A_c - \bar{d}T_y\Delta T_z)^t p = 0,$$

$$T_z x \geqq 0,$$

$$T_y p \geqq 0,$$

where $\bar{d} = \lim_{k\to\infty} d_k \leqq d$. Since the matrix $A_c - \bar{d}T_y\Delta T_z$ is singular and belongs to the interval matrix $[A_c - \bar{d}\Delta, A_c + \bar{d}\Delta]$, we have in light of (5) that $d \leqq \bar{d}$; hence $\bar{d} = d \in [0, 1]$, so that (1)–(4) hold and the proof is complete. □

Next, we formulate some direct consequences of Theorem 2.2. First, we show that there exists a singular matrix in a "normal form."

COROLLARY 2.3. *Let* $[A_c - \Delta, A_c + \Delta]$ *be singular. Then it contains a singular matrix of the form*

$$A_c - dT_y\Delta T_z, \tag{6}$$

where $y, z \in Y$ *and* $d \in [0, 1]$.

Proof. This is an obvious consequence of the assertions (1) and (5) of Theorem 2.2. □

The result can also be given the following geometric formulation.

COROLLARY 2.4. *Let* $[A_c - \Delta, A_c + \Delta]$ *be singular. Then it contains a singular matrix belonging to a segment connecting* A_c *with some vertex of* $[A_c - \Delta, A_c + \Delta]$ *(considered a polyhedron in* R^{n^2}*).*

Proof. For the singular matrix A from (6) we have

$$A = (1 - d)A_c + d(A_c - T_y\Delta T_z),$$

where $d \in [0, 1]$; hence A belongs to the segment connecting A_c with the matrix $A_c - T_y\Delta T_z$, which is a vertex of $[A_c - \Delta, A_c + \Delta]$ since $(A_c - T_y\Delta T_z)_{ij} = (A_c - \Delta)_{ij}$ if $y_iz_j = 1$ and $(A_c - T_y\Delta T_z)_{ij} = (A_c + \Delta)_{ij}$ if $y_iz_j = -1$. □

COROLLARY 2.5. *Let* $[A_c - \Delta, A_c + \Delta]$ *be singular. Then there exists an* $x \neq 0$ *such that*

$$|A_cx| = d\Delta|x|$$

holds, where d is given by (5).

Comment. The assertion is stronger than that of Lemma 2.1; it shows that the inequality holds "uniformly."

Proof. From (1) and (3) we have $|A_cx| = |dT_y\Delta T_zx| = d|\Delta|x|| = d\Delta|x|$. □

3. Real eigenvalues of an interval matrix. In this section we shall apply Theorem 2.2 to study the set of real eigenvalues of an interval matrix $[A_c - \Delta, A_c + \Delta]$ given by

$$L = \{\lambda \in R^1; Ax = \lambda x \text{ for some } A \in [A_c - \Delta, A_c + \Delta], x \neq 0\}.$$

Obviously, L is compact since each $\lambda \in L$ can be written as $\lambda = x'Ax$ for some $A \in [A_c - \Delta, A_c + \Delta]$ and some x with $\|x\|_2 = 1$. In the sequel we shall use the following result.

LEMMA 3.1. $\lambda \in L$ *if and only if the interval matrix*

$$[(A_c - \lambda I) - \Delta, (A_c - \lambda I) + \Delta] \tag{7}$$

is singular.

Proof. If $\lambda \in L$, then $(A - \lambda I)x = 0$ for some $x \neq 0$, where $A - \lambda I$ belongs to the interval matrix (7), which is then singular. Conversely, if (7) is singular, then it contains a matrix A with $Ax = 0$, $x \neq 0$. Then $A + \lambda I \in [A_c - \Delta, A_c + \Delta]$ and $(A + \lambda I)x = \lambda x$; hence $\lambda \in L$. □

We shall first show that each $\lambda \in L$ is an eigenvalue of a matrix of a special form.

THEOREM 3.2. *Let* $\lambda \in L$. *Then there exist* $x \neq 0$, $p \neq 0$, $y, z \in Y$, *and* $d \in [0, 1]$ *such that*

$$(A_c - dT_y \Delta T_z)x = \lambda x, \tag{8a}$$

$$(A_c - dT_y \Delta T_z)^t p = \lambda p, \tag{8b}$$

$$T_z x \geqq 0, \tag{8c}$$

$$T_y p \geqq 0. \tag{8d}$$

Proof. The assertion is a direct consequence of Theorem 2.2 applied according to Lemma 3.1 to the interval matrix (7). □

Let us introduce, as in [6], the matrices

$$A_{yz} = A_c - T_y \Delta T_z$$

for $y, z \in Y$. Obviously, $A_{yz} \in [A_c - \Delta, A_c + \Delta]$ for each $y, z \in Y$.

COROLLARY 3.3. *Let* $\lambda \in L$. *Then* λ *is an eigenvalue of a matrix belonging to a segment connecting* A_c *with some matrix* A_{yz} *for* $y, z \in Y$.

Proof. The proof follows from Theorem 3.2. □

Hence all the real eigenvalues of $[A_c - \Delta, A_c + \Delta]$ are achieved at matrices belonging to a finite number of segments, i.e., to a set of measure zero if $n > 1$; see Hollot and Bartlett [3] for a similar result using the edges of $[A_c - \Delta, A_c + \Delta]$.

Now we shall show that the boundary points of L are eigenvalues of the matrices A_{yz}.

THEOREM 3.4. *Let* $\lambda \in \partial L$. *Then there exist* $x \neq 0$, $p \neq 0$, *and* $y, z \in Y$ *such that*

$$A_{yz} x = \lambda x, \tag{9a}$$

$$A_{yz}^t p = \lambda p, \tag{9b}$$

$$T_z x \geqq 0, \tag{9c}$$

$$T_y p \geqq 0. \tag{9d}$$

Proof. As in the proof of Theorem 2.2, we shall consider separately two cases: (a) $\Delta > 0$ and (b) $\Delta \geqq 0$.

(a) Let $\Delta > 0$. Since L is compact, we have $\partial L \subset L$; hence (8a)–(8d) hold for some $x \neq 0$, $p \neq 0$, $y, z \in Y$ and $d \in [0, 1]$. We shall prove that $d = 1$. Assume to the contrary that $d < 1$. Then from Theorem 3.2 we have

$$|(A_c - \lambda I)x| = d\Delta|x| < \Delta|x|.$$

Hence there exists an $\varepsilon > 0$ such that each $\lambda' \in (\lambda - \varepsilon, \lambda + \varepsilon)$ satisfies

$$|(A_c - \lambda' I)x| < \Delta|x|.$$

Therefore, $[(A_c - \lambda' I) - \Delta, (A_c - \lambda' I) + \Delta]$ is singular according to Lemma 2.1. This implies $\lambda' \in L$ by Lemma 3.1. Hence λ is an interior point of L, which contradicts the assumption that $\lambda \in \partial L$. Thus $d = 1$, so that (8a)–(8d) take on the form of (9a)–(9d).

(b) Let $\Delta \geqq 0$. As in the second part of the proof of Theorem 2.2, for $k = 1, 2, \ldots$ define $\Delta_k = \Delta + (1/k)H$, where H is the matrix of all ones, and let L_k be the set of real eigenvalues of the interval matrix $[A_c - \Delta_k, A_c + \Delta_k]$, so that $L \subset L_k$. For each $k = 1, 2, \ldots$ take a $\lambda_k \in \partial L_k$ that satisfies

$$|\lambda_k - \lambda| = \min\{|\tilde{\lambda} - \lambda|; \tilde{\lambda} \in \partial L_k\}.$$

We shall prove that $\lambda_k \to \lambda$. Assume this is not the case; then there exists an $\varepsilon > 0$ and a subsequence $\{k_j\}$ such that $|\lambda_{k_j} - \lambda| \geqq \varepsilon$ for $j = 1, 2, \ldots$, which in view of the definition of λ_{k_j} gives that $(\lambda - \varepsilon, \lambda + \varepsilon) \subset L_{k_j}$ for each j. Hence for each $\lambda' \in (\lambda - \varepsilon, \lambda + \varepsilon)$ and each $j = 1, 2, \ldots$ there exists a matrix $A_{k_j} \in [A_c - \Delta_{k_j}, A_c + \Delta_{k_j}] \subset [A_c - \Delta_1, A_c + \Delta_1]$ and a vector x_{k_j} with $\|x_{k_j}\|_2 = 1$ such that $A_{k_j}x_{k_j} = \lambda' x_{k_j}$. Taking the limit, we obtain $A_0 x_0 = \lambda' x_0$ for some $A_0 \in [A_c - \Delta, A_c + \Delta]$ and $x_0 \neq 0$. Hence $\lambda' \in L$. This gives $(\lambda - \varepsilon, \lambda + \varepsilon) \subset L$, contrary to $\lambda \in \partial L$. Hence $\lambda_k \to \lambda$. Now, since $\lambda_k \in \partial L_k$ and $\Delta_k > 0$, by applying the result proved in part (a) we obtain that for each $k = 1, 2, \ldots$ there exist $y_k, z_k \in Y$ and vectors x_k, p_k with $\|x_k\|_2 = \|p_k\|_2 = 1$ such that

$$(A_c - T_{y_k}\Delta_k T_{z_k})x_k = \lambda_k x_k,$$

$$(A_c - T_{y_k}\Delta_k T_{z_k})^t p_k = \lambda_k p_k,$$

$$T_{z_k}x_k \geqq 0,$$

$$T_{y_k}p_k \geqq 0$$

hold. Choosing a subsequence along which y_k, z_k remain constant and x_k, p_k converge, we obtain (9a)–(9d), which completes the proof. □

Theorems 3.2 and 3.4 were quite general; to achieve more specific results about the structure of L, we now introduce some assumptions.

Assumption 1. Each $A \in [A_c - \Delta, A_c + \Delta]$ has exactly m real eigenvalues ($1 \leqq m \leqq n$) numbered in such a way that $\lambda_1(A) < \cdots < \lambda_m(A)$.

Then we can define the sets

$$L_i = \{\lambda_i(A); A \in [A_c - \Delta, A_c + \Delta]\}$$

for $i = 1, \ldots, m$.

Assumption 2. $\bar{L}_i \cap \bar{L}_j = \emptyset$ for each $i \neq j$, $i, j \in \{1, \ldots, m\}$ (where the bar denotes closure).

Next we shall assume a sign pattern constancy of the eigenvectors.

Assumption 3. For each $i \in \{1, \ldots, m\}$ there exist vectors $z_i, y_i \in Y$ such that each right eigenvector x (left eigenvector p) pertaining to the ith real eigenvalue of some $A \in [A_c - \Delta, A_c + \Delta]$ satisfies either $T_{z_i}x > 0$ or $T_{z_i}x < 0$ ($T_{y_i}p > 0$ or $T_{y_i}p < 0$).

We formulate the third assumption in this way because $-x$ and $-p$ are also a right eigenvector and left eigenvector, respectively. Notice that $-z_i$ and $-y_i$ also possess the required property.

COROLLARY 3.5. *Let Assumptions* 1–3 *be satisfied, and let* $i \in \{1, \ldots, m\}$. *Then each* $\lambda \in L_i$ *is the ith real eigenvalue of some matrix belonging to the segment connecting* $A_{y_i z_i}$ *with* A_{-y_i, z_i}.

Proof. According to Theorem 3.2 there exist x, p, y, z, and d satisfying (8a)–(8d). Hence λ is an eigenvalue of the matrix $A = A_c - dT_y\Delta T_z$. If $\lambda = \lambda_j(A)$ for some $j \neq i$, then $\lambda \in L_i \cap L_j$, which contradicts Assumption 2. Hence $\lambda = \lambda_i(A)$, which, according to Assumption 3 in conjunction with (8c) and (8d), means that $z = \pm z_i$ and $y = \pm y_i$. Hence either

$$A = A_c - dT_{y_i}\Delta T_{z_i} = \tfrac{1}{2}(1 + d)A_{y_i z_i} + \tfrac{1}{2}(1 - d)A_{-y_i, z_i}$$

or

$$A = A_c - dT_{-y_i}\Delta T_{z_i} = \tfrac{1}{2}(1 - d)A_{y_i z_i} + \tfrac{1}{2}(1 + d)A_{-y_i, z_i}$$

with $d \in [0, 1]$. In both cases A belongs to the segment connecting $A_{y_i z_i}$ with A_{-y_i, z_i}. □

An interval matrix $[A_c - \Delta, A_c + \Delta]$ is called symmetric if both A_c and Δ are symmetric. In this case, if $A \in [A_c - \Delta, A_c + \Delta]$, then $A^t \in [A_c - \Delta, A_c + \Delta]$ also, but generally $[A_c - \Delta, A_c + \Delta]$ can contain nonsymmetric matrices. However, we have the following result.

COROLLARY 3.6. *Let a symmetric interval matrix* $[A_c - \Delta, A_c + \Delta]$ *satisfy Assumptions* 1–3. *Then each* $\lambda \in L$ *is an eigenvalue of some symmetric matrix in* $[A_c - \Delta, A_c + \Delta]$.

Proof. Since each left eigenvector of A_c is also a right eigenvector in this case, it follows from Assumption 3 that $y_i = \pm z_i$ for $i = 1, \ldots, m$. If $\lambda \in L$, then $\lambda \in L_i$ for some i and Corollary 3.5 implies that λ is an eigenvalue of a matrix of the form $A_c - dT_{z_i}\Delta T_{z_i}$ for $d \in [-1, 1]$, which is obviously symmetric. □

Now we are ready to describe the structure of L. The following theorem is a generalization of [7, Thm. 3], where a similar result is proved for interval matrices with Δ of rank one, i.e., of the form $\Delta = qp^t$ for some positive vectors q and p, whereas now Δ can be an arbitrary nonnegative matrix; see also Deif [2].

THEOREM 3.7. *Let an interval matrix* $[A_c - \Delta, A_c + \Delta]$ *satisfy Assumptions* 1–3. *Then*

$$L = \bigcup_{i=1}^{m} L_i,$$

where for each $i \in \{1, \ldots, m\}$ *we have*

$$L_i = [\underline{\lambda}_i, \bar{\lambda}_i] \tag{10}$$

with

$$\underline{\lambda}_i = \min\{\lambda_i(A_{y_i z_i}), \lambda_i(A_{-y_i, z_i})\} \tag{11}$$

and

$$\bar{\lambda}_i = \max\{\lambda_i(A_{y_i z_i}), \lambda_i(A_{-y_i, z_i})\}. \tag{12}$$

Comment. L is thus completely determined by the real components of the spectra of $2m$ matrices $A_{y_i z_i}, A_{-y_i, z_i}$ $(i = 1, \ldots, m)$.

Proof. The assertion for L is simply a consequence of the definition of the L_i's; therefore, we need to prove only (10)–(12). These are obvious if $\Delta = 0$. Assume $\Delta \neq 0$, and let $i \in \{1, \ldots, m\}$, $\lambda \in \partial L_i$. Then $\lambda \in \partial L$, or else Assumption 2 would be violated. Hence, according to Theorem 3.4, λ is an eigenvalue of some A_{yz}, and in view of Assumption 2, $\lambda = \lambda_i(A_{yz})$. From (9c) and (9d) we can infer, as in the proof of Corollary 3.5, that either $\lambda = \underset{\sim}{\lambda}_i$ or $\lambda = \tilde{\lambda}_i$, where we have denoted $\underset{\sim}{\lambda}_i = \lambda_i(A_{-y_i, z_i})$ and $\tilde{\lambda}_i = \lambda_i(A_{y_i z_i})$. Hence L_i has at most two boundary points. Furthermore, let x and p be a right eigenvector and a left eigenvector to $\underset{\sim}{\lambda}_i$ and $\tilde{\lambda}_i$, respectively, such that $T_{z_i}x = |x| > 0$ and $T_{y_i}p = |p| > 0$. Then we have

$$(A_c + T_{y_i}\Delta T_{z_i})x = \underset{\sim}{\lambda}_i x$$

and

$$(A_c - T_{y_i}\Delta T_{z_i})^t p = \tilde{\lambda}_i p.$$

Premultiplying the first equation by p^t and the second by x^t and subtracting, we obtain

$$2|p|^t\Delta|x| = (\underset{\sim}{\lambda}_i - \tilde{\lambda}_i)p^t x.$$

Since $|p| > 0$, $|x| \dot{>} 0$, and $\Delta \neq 0$, this shows that $\underset{\sim}{\lambda}_i \neq \tilde{\lambda}_i$. In a similar manner we can obtain $\underset{\sim}{\lambda}_i \neq \lambda_i(A_c)$ and $\tilde{\lambda}_i \neq \lambda_i(A_c)$. To sum up, we have proved that the compact set L_i consists of at least three different points and has at most two boundary points $\underset{\sim}{\lambda}_i$ and $\tilde{\lambda}_i$; hence L_i is a compact interval whose endpoints are the two boundary points, i.e.,

$$L_i = [\underline{\lambda}_i, \bar{\lambda}_i],$$

where

$$\underline{\lambda}_i = \min\{\underset{\sim}{\lambda}_i, \tilde{\lambda}_i\}$$

and

$$\bar{\lambda}_i = \max\{\underset{\sim}{\lambda}_i, \tilde{\lambda}_i\}. \qquad \square$$

Notice that if the interval matrix is symmetric, then the extremal eigenvalues $\underline{\lambda}_i$, $\bar{\lambda}_i$ are achieved at symmetric matrices $A_{z_i z_i}$ and A_{-z_i, z_i} ($i = 1, \ldots, m$).

The above result has an implication for stability of interval matrices with real eigenvalues. A square matrix is called stable [4] if all its eigenvalues are placed in the open left half of the complex plane. An interval matrix $[A_c - \Delta, A_c + \Delta]$ is called stable if each $A \in [A_c - \Delta, A_c + \Delta]$ is stable. We have the following characterization.

THEOREM 3.8. *Let an interval matrix $[A_c - \Delta, A_c + \Delta]$ satisfy Assumptions 1–3 with $m = n$. Then it is stable if and only if $A_{y_n z_n}$ and A_{-y_n, z_n} are stable.*

Proof. Since $m = n$, all the eigenvalues are real. Therefore, for each $A \in [A_c - \Delta, A_c + \Delta]$ and each $i \in \{1, \ldots, n\}$ we have

$$\lambda_i(A) < \lambda_n(A) \leqq \bar{\lambda}_n = \max\{\lambda_n(A_{y_n z_n}), \lambda_n(A_{-y_n, z_n})\} < 0.$$

Hence A is stable. The "only if" part is obvious. $\square$

4. Real eigenvectors of an interval matrix. In this section we give a characterization of real eigenvectors of matrices belonging to a given interval matrix. In contrast to the eigenvalue case, the situation is much simpler here because it turns out that real eigenvectors can be characterized by a verifiable necessary and sufficient condition.

THEOREM 4.1. *A nonzero real vector x is an eigenvector of some matrix in $[A_c - \Delta, A_c + \Delta]$ if and only if the matrix*

$$X = |x| \cdot |x|^t$$

satisfies

$$(T_z A_c T_z - \Delta) X \leqq X (T_z A_c T_z + \Delta)^t, \tag{13}$$

where the vector z is given by $z_j = 1$ if $x_j \geqq 0$ and $z_j = -1$ otherwise ($j = 1, \ldots, n$).

Proof. For the sake of brevity, let us denote $A_z = T_z A_c T_z$. Then (13) becomes

$$(A_z - \Delta) X \leqq X (A_z + \Delta)^t. \tag{14}$$

"Only if": Let x be a real eigenvector of a matrix $A \in [A_c - \Delta, A_c + \Delta]$, so that $Ax = \lambda x$ for some real λ. Since $T_z x = |x|$ by the definition of z, we have

$$|(A_z - \lambda I)|x|| = |A_c x - \lambda x| = |(A_c - A)x| \leqq \Delta |x|,$$

which implies that

$$(A_z - \Delta)|x| \leqq \lambda |x| \leqq (A_z + \Delta)|x|. \tag{15}$$

Premultiplying the left-hand inequality by $|x|^t$, we obtain

$$(A_z - \Delta)X \leqq \lambda X,$$

and then, transposing the right-hand inequality in (15) and premultiplying the result by $|x|$, we have

$$\lambda X \leqq X(A_z + \Delta)^t,$$

which together give (14).

"If": Conversely, let x be a nonzero real vector such that the matrix $X = |x| \cdot |x|^t$ satisfies (14). Then for each $i, j \in \{1, \dots, n\}$ we have

$$((A_z - \Delta)|x|)_i |x_j| \leqq |x_i|((A_z + \Delta)|x|)_j. \tag{16}$$

Hence for each i, j with $x_i \neq 0$, $x_j \neq 0$ it holds that

$$\frac{((A_z - \Delta)|x|)_i}{|x_i|} \leqq \frac{((A_z + \Delta)|x|)_j}{|x_j|}$$

and, consequently,

$$\max_{x_i \neq 0} \frac{((A_z - \Delta)|x|)_i}{|x_i|} \leqq \min_{x_j \neq 0} \frac{((A_z + \Delta)|x|)_j}{|x_j|}.$$

Hence there exists a λ satisfying

$$\max_{x_i \neq 0} \frac{((A_z - \Delta)|x|)_i}{|x_i|} \leqq \lambda \leqq \min_{x_j \neq 0} \frac{((A_z + \Delta)|x|)_j}{|x_j|}. \tag{17}$$

We shall show that (λ, x) is an eigenpair of some matrix in $[A_c - \Delta, A_c + \Delta]$. Let $k \in \{1, \dots, n\}$. If $x_k \neq 0$, then (17) gives

$$\frac{((A_z - \Delta)|x|)_k}{|x_k|} \leqq \lambda \leqq \frac{((A_z + \Delta)|x|)_k}{|x_k|}.$$

Hence

$$((A_z - \Delta)|x|)_k \leqq \lambda |x_k| \tag{18}$$

and

$$\lambda |x_k| \leqq ((A_z + \Delta)|x|)_k \tag{19}$$

hold. If $x_k = 0$, select an m with $x_m \neq 0$. Then from the inequality (16) applied to $i = k, j = m$ we obtain (18), and similarly applying it to $i = m, j = k$, we get (19). Hence (18) and (19) are valid for each k, which gives that

$$|(A_c - \lambda I)x| = |(A_z - \lambda I)|x|| \leqq \Delta |x|.$$

Then Lemmas 2.1 and 3.1 imply the existence of a matrix $A \in [A_c - \Delta, A_c + \Delta]$ such that $(A - \lambda I)x = 0$. Hence $\lambda \in L$, and x is an eigenvector of A. □

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