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Checking Robust Nonsingularity is NP-Hard*

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Abstract. We consider the following problem: given $k + 1$ square matrices with rational entries, $A_0, A_1, \ldots, A_k$, decide if $A_0 + r_1 A_1 + \cdots + r_k A_k$ is nonsingular for all possible choices of real numbers $r_1, \ldots, r_k$ in the interval [0, 1]. We show that this question, which is closely related to the robust stability problem, is NP-hard. The proof relies on the new concept of *radius of nonsingularity* of a square matrix and on the relationship between computing this radius and a graph-theoretic problem.

Key words. Robustness, NP-complete problems, Robust nonsingularity, Interval matrices.

1. Introduction

It is natural to require that a control system performs satisfactorily even under unknown variations of system parameters in a specified range, i.e., that it is *robust*. The most important performance issue, namely robust *stability*, has been extensively studied recently; we refer to the survey paper by Mansour [M] for a detailed list of references.

In this paper we are concerned with the problem of robust *nonsingularity*. To be more precise, for any two given $n \times n$ matrices A and Δ, with Δ nonnegative, we introduce the *radius of nonsingularity* $d(A, \Delta)$ as the minimum $\varepsilon \geq 0$ for which there exists a singular matrix A' satisfying $A - \varepsilon\Delta \leq A' \leq A + \varepsilon\Delta$. The concept of the radius of nonsingularity is seemingly closely related to Doyle's "structured singular value" introduced in [D2] as a tool for the analysis of feedback systems with structured uncertainties, but we do not pursue this connection in this paper. The concept may also prove useful in the sensitivity analysis of linear systems [D1].

We now summarize the main results. The key result (Theorem 2.1) gives an explicit formula for $d(A, \Delta)$. In order to show that computing $d(A, \Delta)$ is NP-hard, we consider the special case of $\Delta = J$ (the matrix whose all entries are ones) and we show in Theorem 2.2 that

$$d(A, J) = \frac{1}{r(A^{-1})},$$

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where $r(B)$ is defined by

$$r(B) = \max\{z^t By \mid z, y \in \{-1, 1\}^n\}$$

(z^t denotes the transpose of z). Since r is a matrix norm, we first give some upper and lower bounds on it. Then, by establishing a connection between r and the maximum cut in an associated graph, we show that computing $r(B)$ is NP-hard for matrices B with rational entries. As a consequence of the above result we show that the problem of testing singularity of interval matrices is NP-complete. We recall that an interval matrix $A^I = \{A' \mid A - \Delta \le A' \le A + \Delta\}$ is called *singular* if it contains a singular matrix; i.e., A^I is singular if and only if $d(A, \Delta) \le 1$.

Some Notation

We work with square matrices of size $n \times n$ with real entries. We denote by $Q = Q_n$ the n-dimensional discrete cube $Q = \{-1, 1\}^n$, and by e the vector $e = (1, \ldots, 1)^t$. For each $y \in Q$, we denote by T_y the diagonal matrix with the vector y as its diagonal (i.e., $(T_y)_{ii} = y_i$ and $(T_y)_{ij} = 0$ for $i \neq j$). For an arbitrary $n \times n$ matrix A we denote

$$\rho_0(A) = \max\{|\lambda| \mid Ax = \lambda x \text{ for some } x \neq 0, \lambda \text{ real}\},$$

i.e., an analogue of the spectral radius, with maximum being taken only over real eigenvalues; we set $\rho_0(A) = 0$ if no real eigenvalue exists. We use the following matrix norms: $\rho(A) = \sqrt{\rho_0(A^t A)}$ (the spectral norm) and $s(A) = \sum_{i,j} |a_{ij}|$.

2. Radius of Nonsingularity

For an $n \times n$ matrix A and a nonnegative $n \times n$ matrix Δ, we define the *radius of nonsingularity* by

$$d(A, \Delta) = \min\{\varepsilon \ge 0 \mid A - \varepsilon\Delta \le A' \le A + \varepsilon\Delta \text{ for some singular } A'\}.$$

Obviously, $d(A, \Delta) = 0$ if and only if A is singular. On the other hand, it can sometimes be infinite. As an example, consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Delta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here each A' with $A - \varepsilon\Delta \le A' \le A + \varepsilon\Delta$ satisfies $\det A' = -1$, hence $d(A, \Delta)$ is infinite.

Since the case of A singular is trivial, we assume A to be nonsingular in what follows. In this case, using the notation introduced in the previous section, we derive the following explicit formula for $d(A, \Delta)$ (we employ the convention $\frac{1}{0} = \infty$):

Theorem 2.1. *Let A be nonsingular and $\Delta \ge 0$. Then we have*

$$d(A, \Delta) = \frac{1}{\max\{\rho_0(A^{-1}T_y \Delta T_z) \mid y, z \in Q\}}. \tag{1}$$

Proof. First consider the case of $d(A, \Delta)$ finite. For a given $\varepsilon \geq 0$, the existence of a singular matrix A' satisfying $A - \varepsilon\Delta \leq A' \leq A + \varepsilon\Delta$ is equivalent to singularity of the interval matrix $[A - \varepsilon\Delta, A + \varepsilon\Delta]$, which, according to assertion (C3) of Theorem 5.1 in [R], is the case if and only if

$$\rho_0(A^{-1}T_y\varepsilon\Delta T_z) \geq 1$$

holds for some $y, z \in Q$, i.e., if and only if

$$\varepsilon \cdot \max\{\rho_0(A^{-1}T_y\Delta T_z) | y, z \in Q\} \geq 1.$$

Hence the minimum value of ε is given by (1).

If $d(A, \Delta) = \infty$, then by the same result in [R] we have $\varepsilon \cdot \rho_0(A^{-1}T_y\Delta T_z) < 1$ for each $y, z \in Q$ and each $\varepsilon \geq 0$, hence $\rho_0(A^{-1}T_y\Delta T_z) = 0$ for each $y, z \in Q$ and (1) again holds. ■

We show that computing $d(A, \Delta)$ for a given instance A and Δ is NP-hard. For this purpose, we consider the special case $\Delta = J = ee^t$, and we write $d(A)$ instead of $d(A, J)$. We have the following result.

Theorem 2.2. *Let A be nonsingular. Then*

$$d(A) = \frac{1}{r(A^{-1})}, \tag{2}$$

where $r(A^{-1}) = \max\{z^tA^{-1}y | z, y \in Q\}$.

Proof. For $\Delta = ee^t$, we have $A^{-1}T_y\Delta T_z = A^{-1}yz^t$ for each $y, z \in Q$. If λ is a nonzero real eigenvalue of $A^{-1}yz^t$, then from

$$A^{-1}yz^tx = \lambda x$$

we have $z^tx \neq 0$. Premultiplying this equation by z^t and dividing by z^tx gives $\lambda = z^tA^{-1}y$. Thus $\rho_0(A^{-1}yz^t) = |z^tA^{-1}y|$. Then Theorem 2.1 gives $d(A) = 1/r(A^{-1})$, where

$$r(A^{-1}) = \max\{|z^tA^{-1}y| | z, y \in Q\} = \max\{z^tA^{-1}y | z, y \in Q\}. \quad ■$$

The mapping

$$A \mapsto r(A) = \max\{z^tAy | z, y \in Q\}$$

is obviously a matrix norm (i.e., $r(A) \geq 0$, $r(A) = 0$ if and only if $A = 0$, $r(A + B) \leq r(A) + r(B)$ and $r(\lambda A) = |\lambda| r(A)$). Let us note that $r(A)$ has been studied by Brown and Spencer [BS] (see also [ES]) for the case that A is a ± 1-matrix. They proved

$$\sqrt{\frac{2}{\pi}}n^{3/2} < \min\{r(A) | a_{ij} = \pm 1\} < (1 + o(1))n^{3/2} \tag{3}$$

(i.e., the minimum over all ± 1-matrices A). We show in Theorem 2.4 that the lower bound remains valid for any matrix A with $s(A) = n^2$. Since $r(A)$ is a matrix norm, we have $c_1N(A) \leq r(A) \leq c_2N(A)$ for any other norm $N(A)$, where c_1 and c_2 are some constants depending on n only. We present explicit values of such constants for the norms $\rho(A)$ and $s(A)$. Furthermore, we show that computing the exact value of $r(A)$ can be reduced to the max-cut problem in a weighted graph, and, conversely, max-cut can be reduced to computing $r(A)$. The former reduction provides us with a possibility of computing some bounds on $r(A)$ from approximate solutions of max-cut, and the latter implies that computing $r(A)$ is NP-hard.

The next theorem gives a relation between the norms r and ρ. The proof is straightforward and is omitted.

Theorem 2.3. *For every $n \times n$ matrix A we have*

$$\rho(A) \le r(A) \le n\rho(A)$$

and

$$r(A) \ge \sqrt{n\lambda_{\min}(A^t A)}.$$

In the next theorem we compare $r(A)$ with the norm $s(A)$.

Theorem 2.4. *We have*

$$\sqrt{\frac{2}{\pi}}\, n^{-1/2} s(A) \le r(A) \le s(A).$$

Proof. It is well known (see, e.g., the proof of Theorem 15.2 of [ES], or [PRS]) that $E[|e^t y|] \ge \sqrt{2n/\pi}$ for random $y \in Q$. Clearly, $E[|z^t y| \,|\, y \in Q] = E[|e^t y| \,|\, y \in Q]$ for any fixed $z \in Q$. Let $a = (a_1, \dots, a_n)^t$ be a nonnegative vector. Define vectors $a^{(i)} = (a_i, a_{i+1}, \dots, a_n, a_1, \dots, a_{i-1})^t$, $i = 1, \dots, n$, i.e., each $a^{(i)}$ is obtained by a cyclic rotation of a. Set $\alpha = \sum_{i=1}^n a_i$. We have, for a random $y \in Q$,

$$E[|y^t a|] = \frac{1}{n}\sum_{i=1}^n E[|y^t a^{(i)}|] = \frac{1}{n} E\left[\sum_{i=1}^n |y^t a^{(i)}|\right]$$
$$\ge \frac{1}{n} E\left[\left|\sum_{i=1}^n y^t a^{(i)}\right|\right] = \frac{\alpha}{n} E[|e^t y|] \ge \sqrt{\frac{2}{\pi}}\, n^{-1/2}\alpha.$$

Hence, for arbitrary $a \in \mathbb{R}^n$ (not necessarily nonnegative) we have

$$E[|y^t a|] \ge c n^{-1/2}\alpha,$$

where $c = \sqrt{2/\pi}$ and, with A_i denoting the ith row of A,

$$E\left[\sum_{i=1}^n |A_i y|\right] = \sum_{i=1}^n E[|A_i y|] \ge \sum_{i=1}^n c n^{-1/2} \sum_{j=1}^n |a_{ij}| = c n^{-1/2} s(A),$$

hence there exists a $y \in Q$ such that

$$z^t A y = \sum_{i=1}^n |A_i y| \ge c n^{-1/2} s(A),$$

where z is the sign vector of Ay.

The proof of the upper bound is trivial. ∎

Let us note that the original purely probabilistic proof of the lower bound of (3) from [BS] can be modified to an algorithmic one. Thus, for a given ± 1-matrix A, we can construct (in polynomial time) a pair $y, z \in Q$ of vectors such that $z^t A y \geq c n^{3/2}$ where c is the above constant.

In the rest of this section we study a relation between $r(A)$ and the max-cut problem. Again, such a relation is not quite new since the max-cut problem has already been used for reformulation of quadratic optimization problems of type $x^t A x + c^t x$; see, e.g., [B].

The Max-Cut Problem. Let $G = (N, E)$ be a graph and let $c: E \to \mathbb{R}$ be a weight function on edges. The *maximum cut* $\mathrm{mc}(G)$ in the graph G with respect to c is defined as

$$\mathrm{mc}(G) = \max_{S \subset N} c(\delta S),$$

where δS is the set of edges with one endvertex in S and the other one in $N \setminus S$, and $c(F) = \sum_{f \in F} c(f)$ for a subset $F \subset E$.

In order to reduce computing $r(A)$ to the max-cut problem, we define the bipartite graph B_A of a matrix A as the weighted bipartite graph $B_A = (Y \cup Z)$ where Y and Z are two copies of $\{1, \ldots, n\}$ and $E = \{ij \mid a_{ij} \neq 0\}$. The weight of an edge ij is a_{ij}.

Theorem 2.5. *We have* $r(A) = 2\mathrm{mc}(B_A) - e^t A e$.

Proof. Given $y, z \in Q$, define the set

$$S = \{i \in Y \mid y_i = 1\} \cup \{j \in Z \mid z_j = -1\}.$$

We have

$$y^t A z = \sum_{i,j} a_{ij} y_i z_j = \sum_{y_i = z_j} a_{ij} - \sum_{y_i \neq z_j} a_{ij} = 2 \sum_{y_i = z_j} a_{ij} - \sum_{i,j} a_{ij} = 2c(\delta S) - e^t A e,$$

and taking the maximum on both sides gives the result. ∎

The max-cut problem is a known NP-hard problem (see [GJ]). Since it is difficult to find an exact solution, we may use some heuristics. We next survey some of them.

Lower Bounds on Max-Cut.

(i) [PT] If $G = (N, E)$ is a weighted connected graph, then

$$\mathrm{mc}(G) \geq \tfrac{1}{2} + \text{the minimum weight of a spanning tree.}$$

A cut δS satisfying the above inequality can be found in $O(n^3)$ time.

(ii) Lieberherr and Specker have implicitly shown in [LS] the bound

$$\mathrm{mc}(G) \geq c(E) \frac{n}{2n-1}.$$

It is easy to establish the above bound by a probabilistic method. The merit of [LS] is a polynomial-time algorithm achieving it.

Upper Bounds on Max-Cut. An upper bound on max-cut was given by Mohar and Poljak [MP]:

$$\mathrm{mc}(G) \le \frac{n}{4}\lambda_{\max},$$

where $\lambda_{\max}$ is the maximum eigenvalue of the *Laplacian matrix* $L_G = (l_{ij})$ given by

$$l_{ij} = \begin{cases} -c_{ij} & \text{if } ij \in E, \quad i \ne j, \\ 0 & \text{if } ij \notin E, \quad i \ne j, \\ \sum_k c_{ik} & \text{if } i = j. \end{cases}$$

Futher improvement of the eigenvalue bound on the max-cut problem is given by Delorme and Poljak [DP]. Another way to obtain some bounds on the max-cut is via an associated system of linear inequalities; see [DL] for a survey.

We have shown that computing $r(A)$ can be reduced to the max-cut. Now we present an opposite reduction, in order to establish that computing $r(A)$ is NP-hard. We recall that the *cardinality version* of the max-cut, i.e., when all the weights c_{ij} are 0 or 1, is already NP-hard (see problem GT25, p. 196, of [GJ]). The cardinality version is sometimes called the *maximum bipartite subgraph problem.*

Theorem 2.6. *Computing $r(A)$ is NP-hard for a matrix A with rational entries.*

Proof. Let $G = (N, E)$ be a graph. Define a matrix A by

$$a_{ij} = \begin{cases} -1 & \text{if } ij \in E, \quad i \ne j, \\ 0 & \text{if } ij \notin E, \quad i \ne j, \\ M & \text{if } i = j, \end{cases}$$

where M is a sufficiently large integer ($M > 2|E|$ is sufficient). Let $r(A) = z^tAy$ for some $z, y \in Q$. It is easy to see that $z = y$ because of the choice of M. For each $y \in Q$, with $S = \{i | y_i = 1\}$, we have

$$\begin{aligned} y^tAy &= \sum_{i,j} a_{ij}y_iy_j = \sum_{i,j} (-\tfrac{1}{2}a_{ij})((y_i - y_j)^2 - 2) \\ &= -\tfrac{1}{2}\sum_{i,j} a_{ij}(y_i - y_j)^2 + \sum_{i,j} a_{ij} = Mn + 4|\delta S| - 2|E|, \end{aligned}$$

hence $r(A) = Mn + 4\mathrm{mc}(G) - 2|E|$. Thus, an existence of a polynomial-time algorithm to compute $r(A)$ would yield a polynomial-time algorithm to compute $\mathrm{mc}(G)$. Since the latter is an NP-hard problem, computing $r(A)$ is NP-hard as well. ∎

An immediate corollary is the statement formulated in the abstract.

Corollary 2.7. *The following problem is NP-hard.*

Instance: $k + 1$ *square matrices having rational entries*, $A_0, A_1, \ldots, A_k$.
Question: *Is the matrix* $A_0 + r_1A_1 + \cdots + r_kA_k$ *nonsingular for all possible choices of real numbers* $r_1, \ldots, r_k$ *in the interval* $[0, 1]$?

Proof. We show that the problem of computing the radius of nonsingularity can be reduced to it. Assume that we want to decide whether the radius of nonsingularity of a given (rational) matrix A is at least a given $\varepsilon > 0$. Then consider $A_0 = A$, $k = 2n^2$, and define the matrix A_{ij} whose ijth entry is ε and all other entries 0, and $A'_{ij} = -A_{ij}$ for $i, j = 1, \ldots, n$. Clearly, the radius of nonsingularity of A is at least ε if and only if the problem formulated in the corollary has a positive answer. ■

Let us remark that the problem formulated in Corollary 2.7 is algorithmically decidable by the work of Tarski [T]. However, we do not know whether or not the problem belongs to the class NP (though we conjecture that the answer is *yes*). The difficulty arises from the fact that $A_0 + r_1 A_1 + \cdots + r_k A_k$ may be singular only for $(r_1, \ldots, r_k)$ irrational, as shown in the following example. Let $k = 1$, and

$$A_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Then $\det(A_0 + tA_1) = t^2 + 3t + 1$, and $A_0 + tA_1$ is singular if and only if $t = (-3 \pm \sqrt{5})/2$.

Thus, the singularity of $A_0 + r_1 A_1 + \cdots + r_k A_k$ cannot be certified by a direct check for concrete values of $(r_1, \ldots, r_k)$. The method used by Tarski is indirect, based on the Sturm theorem and its generalizations for more variables. However, his certification requires creating a huge family of auxiliary polynomials, and hence it is not polynomial-time bounded in the size of the input data.

Finally, we show that a related problem of singularity of interval matrices is NP-complete. In contrast to Corollary 2.7, we are able to establish the membership in the class NP. A square interval matrix $A^I = \{A' | \underline{A} \le A' \le \overline{A}\}$ is called *singular* if it contains a singular matrix. Consider the decision problem:

Instance: Square interval matrix A^I, where both $\underline{A}$ and $\overline{A}$ are rational matrices.
Question: Is A^I singular?

Theorem 2.8. *The recognition problem of singularity of interval matrices is NP-complete.*

Proof. It is easy to see that computing $r(A)$ can be reduced to the problem whether an interval matrix is singular, and hence the problem is NP-hard. It remains to show that it belongs to the class NP, the class of nondeterministic-polynomial-time problems. We claim that if an interval matrix $A^I = \{A' | \underline{A} \le A' \le \overline{A}\}$ is singular, then there exists a singular matrix A' in the interval such that all the entries of A' are rational numbers whose sizes are bounded by a polynomial in the sizes of the entries of $\underline{A}$ and $\overline{A}$. Such a matrix A' can be "guessed" (i.e., generated by a nondeterministic algorithm), and then it can be checked deterministically in polynomial time that A' is singular, since Gaussian elimination is known to be polynomial time not only in the number of arithmetic operations, but also that the sizes of the numbers that occur during the elimination remain polynomially bounded. (A detailed analysis can be found in [S].) This gives the required nondeterministic-polynomial-time algorithm.

The validity of the claim follows from a result of the second author (see part C7 of Theorem 5.1 of [R]), who proved that if A^I is singular, then there exists a singular matrix $A = (A_{ij}) \in A^I$ with the following property: there is a pair (k, m) of subscripts such that, for every $(i, j) \neq (k, m)$, entry A_{ij} satisfies either $A_{ij} = \underline{A}_{ij}$ or $A_{ij} = \overline{A}_{ij}$. However, since A is singular, the exceptional entry A_{km} can be expressed as a linear combination of subdeterminants of A using the Laplace expansion. Hence the size of all entries of A is bounded by a polynomial in the sizes of entries of $\underline{A}$ and $\overline{A}$. This proves the claim. ■

3. Concluding Comments

We conclude by mentioning two possible applications of the radius of nonsingularity.

Data uncertainty. Assume that we have obtained entries of a matrix $A = (a_{ij})$ as a result of an experiment where the data were measured by a device ensuring some (uniform) precision δ. This means that it is guaranteed that the (unknown) actual value is in the interval $[a_{ij} - \delta, a_{ij} + \delta]$. Now, we have to decide whether A is suitable for further numerical processing, or whether the experiment should be repeated with better precision, which may be more costly. Our decision will depend on whether $\delta \geq d(A)$ (a new experiment is necessary) or $\delta < d(A)$ (the data are sufficiently precise).

Rounding in fixed-point arithmetic. Assume that a matrix A with possibly irrational entries is given. Such a situation may occur when the data are derived formally, e.g., $\sqrt{2}$ may arise as a distance. If we intend to apply a numerical algorithm, we have to round off each entry to some number p of decimal digits. Let $\tilde{A}$ denote the matrix of rounded entries, called a *representation matrix*. If $\|A - \tilde{A}\| > d(\tilde{A})$, this indicates a potentially dangerous situation, since the presence of a singular matrix within the precision of $\tilde{A}$ may mean that $\tilde{A}$ does not reflect well the properties of A.

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