Operations Research Letters 13 (1993) 9–12
North-Holland February 1993

Stability of the optimal basis of a linear program under uncertainty

Jiri Rohn

Faculty of Mathematics and Physics, Charles University, Malostranske nam. 25, 11800 Prague, Czechoslovakia

Received December 1990
Revised August 1992

We prove that the set of optimal basic variables of a linear program remains stable under mutually independent variations of all data within prescribed tolerances if and only if it is stable for a finite subset of explicitly described linear programs from this family. The cardinality of this subset is exponential in the number of constraints.

linear program; uncertainty; optimal basis; stability

Introduction

In this paper we consider a family of linear programming problems

$$\max\{c^{t}x;\ Ax=b,\ x\geqslant 0\} \tag{1}$$

$$A\in A^{\mathrm{I}},\qquad b\in b^{\mathrm{I}},\qquad c\in c^{\mathrm{I}} \tag{2}$$

with A^{I}, b^{I} and c^{I} being given by

$$A^{\mathrm{I}}=\{A;\ A^0-\Delta\leqslant A\leqslant A^0+\Delta\},$$

$$b^{\mathrm{I}}=\{b;\ b^0-\beta\leqslant b\leqslant b^0+\beta\},$$

$$c^{\mathrm{I}}=\{c;\ c^0-\gamma\leqslant c\leqslant c^0+\gamma\},$$

where A^0, $\Delta\in\mathbb{R}^{m\times n}$, b^0, $\beta\in\mathbb{R}^m$, c^0, $\gamma\in\mathbb{R}^n$ and $\Delta\geqslant 0$, $\beta\geqslant 0$, $\gamma\geqslant 0$ (componentwise ordering; i.e., A^{I} is a so-called interval matrix and b^{I}, c^{I} interval vectors) and $m\leqslant n$. This means that (1) and (2) describe a family of LP problems of the form (1) with all the data varying independently of each other within some prescribed tolerances (2). This unifying formulation covers two important classes of problems: (a) LP problems under uncertainty, and (b) LP problems with perturbed data.

The problem (1), (2) can be viewed as a parametric optimization problem ([1,2,4]) in which each entry in the data (A, b, c) is parametrized independently. It is the purpose of this paper to study stability of the optimal solution of (1), (2) in the sense of the following definition (see Murty [3] for the linear programming terminology):

Definition. Let B be an m-tuple of integers from $\{1,\dots,n\}$. We call the problem (1), (2) *B-stable* if each problem (1) with data satisfying (2) has a nondegenerate basic optimal solution with basic variables x_j, $j\in B$, and we call it *strongly B-stable* if, additionally, each such an optimal solution is unique.

It is obvious that this kind of stability is an important issue from a practical point of view because it implies that the qualitative composition of the optimal solution (e.g. the choice of foods in the famous diet problem) remains constant regardless the actual values of the data within given bounds. In this paper we address the problem from a theoretical point of view. We show in the main theorem that the problem (1), (2) is B-stable if and only if a finite subset of explicitly given LP problems of the form (1) with data satisfying (2) have the required property. Unfortunately, the cardinality of this subset is 2^{2m}, hence the result is rather of theoretical interest. Even so, the proof is not quite trivial. This author conjectured an existence of some sort

Correspondence to: Jiri Rohn, Faculty of Mathematics and Physics, Charles University, Malostranske nam. 25, 11800 Prague, Czechoslovakia.

of a finite reduction already in 1979 while writing the paper [5], but was unable to prove it until recently. The impracticality of the result may be overcome by introducing only sufficient stability conditions, but we do not pursue this problem in this paper.

Auxiliary results

In the proof of the main theorem we shall employ three auxiliary results. As two of them are quite recent, we state them here explicitly for convenience of the reader. We introduce the set

$$Y=\{y \in \mathbb{R}^m;\ |y_j|=1 \text{ for } j=1,\dots,m\}$$

and for each $y \in Y$ we denote by T_y the diagonal matrix with diagonal vector y (i.e. $(T_y)_{ii}=y_i$ for each i and $(T_y)_{ij}=0$ for $i \neq j$). Conv X denotes the convex hull of X.

Proposition 1 ([7], Theorem 2). *Let $A \in \mathbb{R}^{m\times n}$, $b \in R^m$ and let for each $y \in Y$ the system of linear inequalities*

$$T_yAx \geqslant T_yb$$

have a solution x_y. Then the system of linear equations

$$Ax=b$$

has a solution in $\operatorname{Conv}\{x_y;\ y \in Y\}$.

Next we have two results concerning P-matrices. Let us recall that a square matrix is called a P-matrix if all its principal minors are positive [3].

Proposition 2 ([6], Theorem 1.2). *Let $\tilde{A}^{\mathrm{I}}$ be a square interval matrix such that each $A \in \tilde{A}^{\mathrm{I}}$ is nonsingular. Then $A_1^{-1}A_2$ is a P-matrix for each A_1, $A_2 \in \tilde{A}^{\mathrm{I}}$.*

For a given $p=(p_i) \in \mathbb{R}^m$ we define $p^+=(p_i^+)$, $p^-=(p_i^-)$ and $|p|=(|p_i|)$, where $p_i^+=\max\{p_i, 0\}$, $p_i^-=\max\{-p_i, 0\}$ $(i=1,\dots,m)$. Then we have $p=p^+-p^-$, $|p|=p^++p^-$. The following result is well known (Murty [3]):

Proposition 3. *Let M be a P-matrix. Then the linear complementarity problem*

$$p^+=Mp^-+q$$

has a unique solution for each right-hand side vector q.

Main result

Let $B=(B_1,\dots,B_m)$ be an m-tuple of mutually different integers from $\{1,\dots,n\}$. As customary in the linear programming theory, for each $A \in A^{\mathrm{I}}$ we denote by A_B the square submatrix of A consisting of columns with indices $B_1,\dots,B_m$, and by A_N the remaining (nonbasic) part of A. A similar notation also applies to vectors: c_B, x_B, c_N, γ_N etc. In addition to the set $Y \subset \mathbb{R}^m$ defined above we introduce also the set

$$Z_B=\{z \in \mathbb{R}^n;\ |z_j|=1 \text{ for } j \in B,$$
$$z_j=1 \text{ for } j \notin B,\ j=1,\dots,n\},$$

so that Z_B has 2^m elements. We extend our notation T_z also to vectors from Z_B (in this case it denotes an $n \times n$ matrix). For each $y \in Y$, $z \in Z_B$ we define the matrix

$$A_{yz}=A^0-T_y\Delta T_z,$$

and the vectors

$$b_y=b^0+T_y\beta,$$

$$c_z=c^0+T_z\gamma.$$

Notice that for each i, j we have $(A_{yz})_{ij}=(A^0-\Delta)_{ij}$ if $y_iz_j=1$ and $(A_{yz})_{ij}=(A^0+\Delta)_{ij}$ otherwise, hence each coefficient of A_{yz} is fixed either at the lower, or at the upper bound of the respective entry, and an analogue holds for b_y, c_z. Our characterization of B-stability is formulated in terms of these quantities:

Theorem. *The problem* (1), (2) *is* [*strongly*] *B-stable if and only if for each $y \in Y$, $z \in Z_B$ the* LP *problem*

$$\max\{c_z^{\mathrm{t}}x;\ A_{yz}x=b_y,\ x \geqslant 0\} \tag{3}$$

has a [*unique*] *nondegenerate basic optimal solution with basic variables x_j, $j \in B$.*

Comment. Notice that each LP problem of the form (3) is a 'vertex problem' since its data (A, b, c) form a vertex of the set $A^{\mathrm{I}} \times b^{\mathrm{I}} \times c^{\mathrm{I}}$ (considered a rectangle in $\mathbb{R}^{mn+m+n}$). Not all the 'vertex problems' are included, however, since their number in the worst case $\Delta > 0$, $\beta > 0$, $\gamma > 0$ is 2^{mn+m+n} while there are 2^{2m} problems of type (3).

Proof. The 'only if' part is obvious since $A_{yz} \in A^I$, $b_y \in b^I$ and $c_z \in c^I$ for each $y \in Y$, $z \in Z_B$. In the proof of the 'if' part we shall show that for each $A \in A^I$, $b \in b^I$, $c \in c^I$ there holds: (i) A_B is nonsingular, (ii) $A_B x_B = b$ has a positive solution, (iii) the solution of the equation $A_B^t p = c_B$ satisfies $A_N^t p \geq c_N$. This will prove that x (with the basic part x_B from (ii) and nonbasic part zero) is a nondegenerate basic optimal solution of (1) with basis index set B (p is then the dual optimal solution). Thus we are confined to prove (i)–(iii).

(i) Let $y \in Y$ and $z = (1, 1, \ldots, 1)^t \in Z_B$. Since (3) has a nondegenerate basic optimal solution with basic variables x_j, $j \in B$, we have that the system

$$(A_{yz})_B x = b_y \tag{4}$$

has a positive solution $x_y \in \mathbb{R}^m$. Rearranging (4), we see that x_y satisfies

$$T_y\left(A_B^0 x_y - b^0\right) = \Delta_B x_y + \beta.$$

Now let $A \in A^I$ and $b \in b^I$. Since $|T_y(A_B - A_B^0)x_y| \leq \Delta_B x_y$ and $|T_y(b^0 - b)| \leq \beta$, we obtain

$$\begin{aligned} T_y(A_B x_y - b) &= T_y\left(A_B^0 x_y - b^0\right) \\ &\quad + T_y\left(A_B - A_B^0\right)x_y + T_y(b^0 - b) \\ &\geq \Delta_B x_y + \beta - \Delta_B x_y - \beta = 0, \end{aligned}$$

hence

$$T_y A_B x_y \geq T_y b \tag{5}$$

holds for each $y \in Y$, which in view of Proposition 1 means that $A_B x = b$ has a solution for each $A \in A^I$, $b \in b^I$. Now consider two cases: (a) If $\beta > 0$, then b^I contains m linearly independent vectors, therefore A_B is nonsingular. (b) If $\beta_i = 0$ for some i, take a perturbed vector $\beta' = \beta + (\varepsilon, \varepsilon, \ldots, \varepsilon)^t$ where $\varepsilon > 0$ is chosen so that each of the finitely many systems

$$(A_{yz})_B x = b_y + T_y(\varepsilon, \varepsilon, \ldots, \varepsilon)^t \tag{4'}$$

preserve a positive solution (this is possible due to nonsingularity of $(A_{yz})_B$ and nondegeneracy). Since $\beta' > 0$, the previous argument applies to prove again that A_B is nonsingular.

(ii) As we have seen in (i), the inequality (5) holds for each $A \in A^I$, $b \in b^I$ and $y \in Y$, hence according to Proposition 1 the unique solution of the equation $A_B x_B = b$ belongs to $\mathrm{Conv}\{x_y;\ y \in Y\}$ and therefore is positive since each x_y is positive.

(iii) To prove the last assertion, we shall first show that for each $z \in Y$ the nonlinear equation

$$T_z\left(\left(A_B^0\right)^t p - c_B^0\right) = \Delta_B^t |p| + \gamma_B \tag{6}$$

has a unique solution. In fact, using $p = p^+ - p^-$ and $|p| = p^+ + p^-$, (6) can be equivalently rearranged to the form

$$\begin{aligned} p^+ &= \left[\left(A_B^0\right)^t - T_z \Delta_B^t\right]^{-1}\left[\left(A_B^0\right)^t + T_z \Delta_B^t\right] p^- \\ &\quad + \left[\left(A_B^0\right)^t - T_z \Delta_B^t\right]^{-1}\left(c_B^0 + T_z \gamma_B\right) \end{aligned} \tag{7}$$

which is a linear complementarity problem whose matrix is a P-matrix due to Proposition 2 since both the matrices $(A_B^0)^t - T_z \Delta_B^t$ and $(A_B^0)^t + T_z \Delta_B^t$ belong to the interval matrix $\{A_B^t;\ A \in A^I\}$ whose elements are all nonsingular according to (i). Hence Proposition 3 gives that (7), and thus also (6), has a unique solution p_z. Now, let $A \in A^I$ and $c \in c^I$. We shall first prove that

$$A_N^t p_z \geq c_N \tag{8}$$

holds for each $z \in Y$. To this end, for a given $z \in Y$ define a $y \in Y$ as follows: $y_i = 1$ if $(p_z)_i \geq 0$ and $y_i = -1$ otherwise ($i = 1, \ldots, m$). Then $|p_z| = T_y p_z$ and substituting into (6) we obtain

$$\left(A_B^0 - T_y \Delta_B T_z\right)^t p_z = c_B^0 + T_z \gamma_B,$$

which can be written as

$$\left(A_{yz^*}\right)_B^t p_z = (c_{z^*})_B,$$

where the vector $z^* \in Z_B$ is defined by $z_B^* = z$ and $z_N^* = (1, 1, \ldots, 1)^t$. Then the complementary slackness condition of linear programming gives that p_z is the unique dual optimal solution to the problem (3) for y, z^*, hence

$$\left(A_{yz^*}\right)_N^t p_z \geq (c_{z^*})_N \tag{9}$$

holds and since $|(A_N - A_N^0)^t p_z| \leq \Delta_N^t |p_z|$, we have

$$\begin{aligned} A_N^t p_z &= \left(A_N^0\right)^t p_z + \left(A_N - A_N^0\right)^t p_z \\ &\geq \left(A_N^0\right)^t p_z - \Delta_N^t |p_z| \\ &= \left(A^0 - T_y \Delta\right)_N^t p_z = \left(A_{yz^*}\right)_N^t p_z \geq (c_{z^*})_N \\ &= c_N^0 + \gamma_N \geq c_N, \end{aligned}$$

which proves (8). Now, let p be the solution to $A_B^t p = c_B$ (which is unique due to (i)). In a similar way as in the part (i) we can prove from (6) that

$$T_z A_B^t p_z \geq T_z c_B$$

holds for each $z \in Y$, hence Proposition 1 gives that $p = \Sigma_{z \in Y} \lambda_z p_z$ for some real numbers $\lambda_z \in [0, 1]$, $z \in Y$, satisfying $\Sigma_{z \in Y} \lambda_z = 1$, thus from (8) we finally obtain

$$A_N^t p = \sum_{z \in Y} \lambda_z A_N^t p_z \geqslant \sum_{z \in Y} \lambda_z c_N = c_N, \tag{10}$$

which concludes the proof of part (iii).

If, moreover, each problem (3) has a unique optimal solution, then the inequality (9), and consequently also (8) and (10), hold sharply which means that each problem (1) under (2) has a unique nondegenerate basic optimal solution, hence the problem (1), (2) is strongly B-stable. □

References

[1] W. Dinkelbach, *Sensitivitätsanalyse und parametrische Optimierung*, Springer-Verlag, Berlin, 1969.

[2] T. Gal, *Postoptimal Analyses, Parametric Programming and Related Topics*, McGraw-Hill, New York, 1979.

[3] K. Murty, *Linear and Combinatorial Programming*, Wiley, New York, 1976.

[4] F. Nožička, J. Guddat, H. Hollatz and B. Bank, *Theorie der linearen parametrischen Optimierung*, Akademie Verlag, Berlin, 1974.

[5] J. Rohn, "Strong solvability of interval linear programming problems", *Computing* **26**, 79–82 (1981).

[6] J. Rohn, "Systems of linear interval equations", *Linear Algebra Appl.* **126**, 39–78 (1989).

[7] J. Rohn, "An existence theorem for systems of linear equations", *Linear and Multilinear Algebra* **29**, 141–144 (1991).