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Short communication

A note on solvability of a class of linear complementarity problems

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We give a characterization of unique solvability of an infinite family of linear complementarity problems of a special form by means of a finite subset of this family.

Key words: Linear complementarity problem, nonsingular matrix, P-matrix.

A linear complementarity problem is a problem of the form

$$y = Mz + q$$

$$y \geqslant 0, \quad z \geqslant 0,$$

$$y^{\mathrm{T}} z = 0,$$

where M is an $n \times n$ matrix and q an n-dimensional vector; we shall denote this problem by LCP(M, q). A detailed exposition of the linear complementarity theory may be found in Murty's book [1]. In this short note we apply some recent results on systems of linear equations with inexact data [2] to obtain some necessary and sufficient conditions for unique solvability of a whole class of linear complementarity problems of the form LCP($M_1^{-1} M_2$, q) with $A \leqslant M_1 \leqslant B$ and $A \leqslant M_2 \leqslant B$, where A and B are two given $n \times n$ matrices and $q \in \mathbb{R}^n$. (Here, as in the sequel, matrix and vector inequalities are understood componentwise and the inverse of a matrix M is assumed to exist whenever the symbol M^{-1} is used.)

Before formulating the main result we introduce some notations. A diagonal matrix S is said to be a signature matrix if each its diagonal element is equal to 1 or -1, clearly there are 2^n signature matrices of size n, among them the unit matrix I. Let A, B be two $n \times n$ matrices, $A \leqslant B$, and let S be a signature matrix of the same size. We introduce the matrix

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$$M_S = K_S^{-1} L_S$$

where

$$K_S = \tfrac{1}{2}(I+S)A + \tfrac{1}{2}(I-S)B$$

and

$$L_S = \tfrac{1}{2}(I-S)A + \tfrac{1}{2}(I+S)B\ .$$

Since S is a signature matrix, each element of K_S is equal to the respective element of either A or B, which implies $A \leqslant K_S \leqslant B$; the same holds for L_S. Further let

$$q_S = K_S^{-1} Se$$

where $e = (1, 1, \ldots, 1)^{\mathrm{T}}$. Let us recall that a square matrix is called a P-matrix if all its principal minors are positive.

Now we have this result:

Theorem. *Let A, B be two $n \times n$ matrices, $A \leqslant B$. Then the following assertions are equivalent*:

(i) *Each matrix C satisfying $A \leqslant C \leqslant B$ is nonsingular.*

(ii) *The* LCP$(M_1^{-1}M_2, q)$ *has a unique solution for all matrices M_1, M_2 satisfying $A \leqslant M_1 \leqslant B$, $A \leqslant M_2 \leqslant B$ and each right-hand side vector q.*

(iii) *The* LCP(M_S, q_S) *has a solution for each signature matrix S.*

(iv) *The system*

$$y = M_S z + q_s, \qquad (1)$$

$$y \geqslant 0, \quad z \geqslant 0,$$

has a solution for each signature matrix S.

(v) *M_s is a P-matrix for each signature matrix S.*

Proof. (i) ⇒ (ii), (i) ⇒ (v): If (i) holds, then according to Theorem 1.2 in [2], each matrix of the form $M_1^{-1}M_2$, where $A \leqslant M_1 \leqslant B$ and $A \leqslant M_2 \leqslant B$, is a P-matrix. This proves (v) due to the definition of M_S and also implies (ii) in view of the well-known result on unique solvability of a linear complementarity problem LCP(M, q) with a P-matrix M, see [1].

(ii) ⇒ (iii) follows from the fact that M_S is of the form $M_S = K_S^{-1} L_S$, where $A \leqslant K_S \leqslant B$ and $A \leqslant L_S \leqslant B$.

(iii) ⇒ (iv) is obvious since the solution of LCP(M_S, q_S) also solves the system (1).

(iv) ⇒ (i): If y, z solve (1), then they satisfy the system

$$K_S y - L_S z = Se, \qquad (2)$$

$$y \geqslant 0, \quad z \geqslant 0.$$

According to the assertion (A2) of Theorem 5.1 in [2], the existence of a solution to a system (2) for an arbitrary signature matrix S implies the nonsingularity of each matrix C satisfying $A \leqslant C \leqslant B$.

(v) ⇒ (i): Follows from the assertion (B1) of Theorem 5.1 in [2]. □

The merit of this result is the fact that unique solvability of an infinite family of linear complementarity problems

$$\begin{aligned} &\mathrm{LCP}(M_1^{-1}M_2, q), \\ &A \leqslant M_1 \leqslant B, \\ &A \leqslant M_2 \leqslant B, \\ &q \in \mathbb{R}^n, \end{aligned} \tag{3}$$

can be characterized by means of a finite subset of this family (equivalence (ii) ⇔ (iii)). But even more, as the assertion (iv) shows, the existence of nonnegative solutions to a finite number of systems of linear equations of the type (1) (where the complementarity constraint is dropped) is sufficient for unique solvability of each problem in the family (3); however, the number of test problems is exponential in matrix size. Nevertheless, there exists a verifiable sufficient condition: if

$$\rho(\,|2I - Q(A+B)\,| + |Q|\,(B-A)\,) < 2$$

holds for some (but arbitrary) $n \times n$ matrix Q (where ρ is the spectral radius and $|\cdot|$ denotes the absolute value of a matrix), then each matrix C satisfying $A \leqslant C \leqslant B$ is nonsingular [3], hence each problem in the family (3) is uniquely solvable. As explained in [3], for practical verification it is recommended to choose Q as the computed value of $(\frac{1}{2}(A+B))^{-1}$. Notice also that if (3) contains a problem which is not uniquely solvable, then there exists a signature matrix S such that either K_S is singular, or LCP(M_S, q_S) does not possess a solution (assertion (iii)).

Linear complementarity problems of the form LCP($M_1^{-1}M_2$, q), $A \leqslant M_1 \leqslant B$, $A \leqslant M_2 \leqslant B$ arise naturally in solving systems of linear equations with inexact data; see [2] for details.

References

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