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POSITIVE DEFINITENESS AND STABILITY OF INTERVAL MATRICES*

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Abstract. Characterizations of positive definiteness, positive semidefiniteness, and Hurwitz and Schur stability of interval matrices are given. First it is shown that an interval matrix has some of the four properties if and only if this is true for a finite subset of explicitly described matrices, and some previous results of this type are improved. Second it is proved that a symmetric interval matrix is positive definite (Hurwitz stable, Schur stable) if and only if it contains at least one symmetric matrix with the respective property and is nonsingular (for Schur stability, two interval matrices are to be nonsingular). As a consequence, verifiable sufficient conditions are obtained for positive definiteness and Hurwitz and Schur stability of symmetric interval matrices.

Key words. interval matrix, positive definiteness, positive semidefiniteness, Hurwitz stability, Schur stability, nonsingularity

AMS subject classifications. 15A18, 15A48, 65G10, 93D09

Introduction. In this paper we study positive definiteness, positive semidefiniteness, and Hurwitz and Schur stability of square interval matrices defined in the following way: an interval matrix A^I is said to be positive definite (positive semidefinite, Hurwitz stable) if each matrix $A \in A^I$ is positive definite (positive semidefinite, Hurwitz stable); a slight deviation from this definition is made for Schur stability where a symmetric interval matrix A^I is said to be Schur stable if each *symmetric* $A \in A^I$ is Schur stable. Positive (semi)definiteness of interval matrices is studied in § 2, Hurwitz stability in § 3, and Schur stability in § 4. There are two main streams of results that run across these sections.

First, we show that for each of the four properties listed it holds that A^I (assumed to be symmetric in stability cases) has the property if and only if this is true for a finite subset of explicitly described matrices in A^I. The result for positive (semi)definiteness is given in Theorem 2, where the respective subset is shown to be of cardinality 2^{n-1} (in the worst case) for an $n \times n$ interval matrix A^I; this theorem improves considerably the earlier result by Shi and Gao [13], which used $2^{n(n-1)/2}$ test matrices. A similar result is given in Theorem 6 for Hurwitz stability of symmetric interval matrices, which is again characterized by a subset of matrices of cardinality 2^{n-1}. Hertz [6] has recently proved that stability of this subset implies stability of each symmetric matrix in A^I; our result shows that stability of this subset already implies stability of the whole of A^I.

Second, we show that a symmetric interval matrix A^I is positive definite (Hurwitz stable, Schur stable) if and only if it contains at least one symmetric matrix with the respective property and is regular (for Schur stability, two associated interval matrices are to be regular; A^I is called regular [9] if each $A \in A^I$ is nonsingular). These results, proved in Theorems 3, 8, and 11, reduce the number of test matrices to one but do not remove exponentiality from the verification process because all the necessary and sufficient regularity conditions known ([9], [12]) employ some subset of test matrices whose cardinality is exponential in the matrix size. Nevertheless, because there exists a sufficient regularity condition due to Beeck [2], which is known to cover most practical examples, employing it in the above characterizations leads to sufficient conditions for positive definiteness and Hurwitz and Schur stability of symmetric interval matrices (Theorems 4, 9, and 12), which can be expected to work well in practical cases. In the final remark in § 5, we give a modification of the Beeck's condition that enables us to use an approx-

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imation of the inverse of the center matrix of A^I instead of the exact inverse as required in the original formulation.

1. Notations and auxiliary results. We introduce some notations and prove a theorem that sums up the basic technical results to be used later in the proofs of the main theorems.

For a square real matrix $A = (a_{ij})$, we denote the transpose by A^T, the spectral radius by $\rho(A)$, and we introduce its absolute value as the matrix $|A| = (|a_{ij}|)$. A matrix A is called symmetric if $A = A^T$. Symmetric matrices are known to have all eigenvalues real; we shall denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, the minimum and maximum eigenvalue of A, respectively (obviously, $\lambda_{\min}(-A) = -\lambda_{\max}(A)$). Matrix inequalities, as $A \leqq B$ or $A < B$, are to be understood componentwise.

Let A_c and Δ be real $n \times n$ matrices, $\Delta \geqq 0$. The set of matrices

$$A^I = [A_c - \Delta, A_c + \Delta] = \{A; A_c - \Delta \leqq A \leqq A_c + \Delta\}$$

is called an interval matrix. A^I is said to be symmetric if both A_c and Δ are symmetric. With each interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ we shall associate the symmetric interval matrix

$$A_s^I = [A_c' - \Delta', A_c' + \Delta'],$$

where A_c' and Δ' are given by

$$A_c' = \tfrac{1}{2}(A_c + A_c^T)$$

and

$$\Delta' = \tfrac{1}{2}(\Delta + \Delta^T).$$

Obviously, if $A \in A^I$, then $\frac{1}{2}(A + A^T) \in A_s^I$ and A^I is symmetric if and only if $A^I = A_s^I$.

We introduce an auxiliary index set

$$Y = \{z \in R^n;\ |z_j| = 1 \quad \text{for } j = 1, \ldots, n\},$$

i.e., Y is the set of all ± 1-vectors; hence, its cardinality is 2^n. For each $z \in Y$ we shall denote by T_z the $n \times n$ diagonal matrix with diagonal vector z. Now for each $z \in Y$ let us define the matrix A_z by

$$A_z = A_c - T_z \Delta T_z.$$

Then for each i, j we have $(A_z)_{ij} = (A_c)_{ij} - z_i \Delta_{ij} z_j = (A_c - \Delta)_{ij}$ if $z_i z_j = 1$ and $(A_z)_{ij} = (A_c + \Delta)_{ij}$ if $z_i z_j = -1$; hence, $A_z \in A^I$ for each $z \in Y$ and because $A_{-z} = A_z$, the number of mutually different matrices A_z is at most 2^{n-1} (and equal to 2^{n-1} if $\Delta > 0$). If A^I is symmetric, then each A_z is symmetric. The matrices A_z, $z \in Y$, will be used in § 2 to characterize positive (semi)definiteness of an interval matrix by finite means.

Let us now introduce a function $f: R^{n \times n} \to R^1$ defined for a matrix $A \in R^{n \times n}$ by

$$f(A) = \min_{x \neq 0} \frac{x^T A x}{x^T x}. \tag{1}$$

Obviously, f is well defined. In the following theorem we sum up the basic properties of f that will be used in the proofs of the main theorems in the subsequent sections.

THEOREM 1. *The function f has the following properties:*

(i) $f(A) = f(\frac{1}{2}(A + A^T))$ *for each* $A \in R^{n \times n}$;

(ii) $f(A) = \lambda_{\min}(A)$ *for each symmetric* $A \in R^{n \times n}$;

(iii) $|f(A + D) - f(A)| \leqq \rho(\frac{1}{2}(D + D^T))$ *for each* $A, D \in R^{n \times n}$;

(iv) *f is continuous in $R^{n\times n}$;*

(v) *for each interval matrix A^I we have*

$$\min\{f(A); A\in A^I\} = \min\{f(A_z); z\in Y\};$$

(vi) *for each interval matrix A^I we have*

$$\min\{f(A); A\in A^I\} = \min\{f(A); A\in A_s^I\};$$

(vii) *each interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ satisfies*

$$\min\{f(A); A\in A^I\} \geqq f(A_c) - \rho(\Delta');$$

(viii) *if A is symmetric and $f(A) = 0$, then A is singular.*

Proof. (i) follows from the fact that $x^TAx = x^T(\frac{1}{2}(A+A^T))x$ for each $A\in R^{n\times n}$ and $x\in R^n$. (ii) is well known (cf., e.g., Parlett [10]). To prove (iii), first observe that from (1) it follows

$$f(A+D)\geqq f(A)+f(D)$$

for each A and D; this inequality implies

$$f(A) = f((A+D)+(-D)) \geqq f(A+D)+f(-D),$$

which together gives

$$\begin{aligned}|f(A+D)-f(A)| &\leqq \max\{|f(D)|, |f(-D)|\}\\ &= \max\{|f(\tfrac{1}{2}(D+D^T))|, |f(-\tfrac{1}{2}(D+D^T))|\}\\ &= \max\{|\lambda_{\min}(\tfrac{1}{2}(D+D^T))|, |\lambda_{\max}(\tfrac{1}{2}(D+D^T))|\}\\ &= \rho(\tfrac{1}{2}(D+D^T)).\end{aligned}$$

For (iv) take a matrix norm $\|\cdot\|$ such that $\|A^T\| = \|A\|$ for each A. Then from (iii) we obtain

$$|f(A+D)-f(A)| \leqq \|\tfrac{1}{2}(D+D^T)\| \leqq \|D\|$$

for each A and D, which proves that f is continuous in $R^{n\times n}$.

To prove (v), let $A\in A^I$ and $x\neq 0$. Because $|x^T(A-A_c)x| \leqq |x|^T\Delta|x|$, we obtain $x^TAx = x^TA_cx + x^T(A-A_c)x \geqq x^TA_cx - |x|^T\Delta|x|$. Define a $z\in Y$ as follows: $z_j = 1$ if $x_j\geqq 0$ and $z_j = -1$ otherwise ($j = 1,\dots,n$), then $|x| = T_zx$ and we have

$$x^TAx \geqq x^TA_cx - x^TT_z\Delta T_zx = x^TA_zx;$$

hence,

$$\frac{x^TAx}{x^Tx} \geqq \frac{x^TA_zx}{x^Tx} \geqq f(A_z) \geqq \min\{f(A_z); z\in Y\},$$

which implies that

$$f(A)\geqq \min\{f(A_z); z\in Y\}$$

holds for each $A\in A^I$ and because $A_z\in A^I$ for each $z\in Y$, the assertion follows.

To prove (vi), for each $z\in Y$ denote by A_z' the matrix A_z for A_s^I, i.e.,

$$A_z' = A_c' - T_z\Delta' T_z = \tfrac{1}{2}(A_c + A_c^T) - T_z(\tfrac{1}{2}(\Delta+\Delta^T))T_z = \tfrac{1}{2}(A_z + A_z^T).$$

Then employing (i) we obtain

$$f(A_z) = f(\tfrac{1}{2}(A_z + A_z^T)) = f(A_z');$$

hence, the assertion (v) implies that the minimum values of f over A^I and A_s^I are equal.

For (vii) let $A \in A^I$. Since $|A - A_c| \leqq \Delta$, using (iii) and Proposition 3.2.4 in [9] we obtain $|f(A) - f(A_c)| \leqq \rho(\frac{1}{2}(A - A_c) + \frac{1}{2}(A - A_c)^T) \leqq \rho(\frac{1}{2}(\Delta + \Delta^T)) = \rho(\Delta')$, which gives $f(A) \geqq f(A_c) - \rho(\Delta')$ and thus also

$$\min \{f(A); A \in A^I\} \geqq f(A_c) - \rho(\Delta').$$

For (viii) under the assumptions, zero is an eigenvalue of A due to (ii), hence A is singular. □

2. Positive (semi)definiteness of interval matrices. A square (not necessarily symmetric) matrix A is called positive semidefinite if $f(A) \geqq 0$, which, in view of (1), means that $x^TAx \geqq 0$ for each x (hence our definition conforms to the usual one). Similarly, A is said to be positive definite if $f(A) > 0$ (i.e., $x^TAx > 0$ for each $x \neq 0$). An *interval matrix* A^I is said to be positive (semi)definite if each $A \in A^I$ is positive (semi)definite. As a consequence of Theorem 1 we obtain this characterization.

THEOREM 2. *Let A^I be a square interval matrix. Then the following assertions are equivalent:*

(a) *A^I is positive (semi)definite,*

(b) *A_s^I is positive (semi)definite,*

(c) *A_z is positive (semi)definite for each $z \in Y$.*

Proof. We shall prove the theorem for the case of positive definiteness of A^I; the proof for positive semidefiniteness runs quite analogously. By definition, A^I is positive definite if and only if

$$\min \{f(A); A \in A^I\} > 0$$

holds. Then the equivalence of (a) and (b) follows from the assertion (vi) of Theorem 1 and that of (a) and (c) from the assertion (v) of the same theorem. □

The assertion (c) shows that positive (semi)definiteness of an interval matrix can be verified by testing 2^{n-1} matrices from A^I for positive (semi)definiteness. Hence, this theorem improves considerably the earlier result by Shi and Gao [13], which required testing $2^{n(n-1)/2}$ matrices from A^I (the so-called vertex matrices) for positive (semi)definiteness; moreover, their result was given for symmetric interval matrices only. We note that Białas and Garloff [4] proved a similar characterization of interval P-matrices (each $A \in A^I$ is a P-matrix if and only if each A_z, $z \in Y$ is a P-matrix), although they did not explicitly use the matrices A_z.

The equivalence "$(a) \Leftrightarrow (b)$" reveals another important property, namely that verification of positive (semi)definiteness of A^I always can be performed by inspecting the associated symmetric interval matrix A_s^I; hence, we can restrict our attention in the sequel to symmetric interval matrices only. First we have this corollary.

COROLLARY. *Let a symmetric interval matrix A^I be positive semidefinite. Then it is positive definite if and only if all the matrices A_z, $z \in Y$ are nonsingular.*

Proof. The "only if" part is obvious because each positive definite matrix is nonsingular. To prove the "if" part, assume to the contrary that A^I is positive semidefinite but not positive definite. Then from the assertion (c) of Theorem 2 it follows that there exists a matrix A_z that is positive semidefinite but not positive definite. Then $f(A_z) = 0$ and because A_z is symmetric, we have that A_z is singular (Theorem 1, (viii)), which is a contradiction. □

In the next theorem we prove that positive definiteness of symmetric interval matrices is closely related to regularity. Let us recall that a square interval matrix A^I is called regular [9] if each $A \in A^I$ is nonsingular.

THEOREM 3. *A symmetric interval matrix A^I is positive definite if and only if it is regular and contains at least one positive definite matrix.*

Proof. Again, the "only if" part is obvious. In the proof of the "if" part, assume to the contrary that A^I is regular and contains a positive definite matrix A_0 but is not positive definite, so that $x^T A_1 x \leqq 0$ for some $A_1 \in A^I$ and $x \neq 0$. Define $\tilde{A}_0 = \frac{1}{2}(A_0 + A_0^T)$ and $\tilde{A}_1 = \frac{1}{2}(A_1 + A_1^T)$, then both $\tilde{A}_0$ and $\tilde{A}_1$ are symmetric, belong to A^I, and satisfy

$$f(\tilde{A}_0) = f(A_0) > 0$$

and

$$f(\tilde{A}_1) = f(A_1) \leqq 0.$$

Now define a real function φ of one real variable by

$$\varphi(t) = f(t\tilde{A}_0 + (1-t)\tilde{A}_1), \qquad t \in [0, 1].$$

Then φ is continuous by the assertion (iv) of Theorem 1 and because $\varphi(0)\varphi(1) = f(\tilde{A}_1)f(\tilde{A}_0) \leqq 0$, there exists a $t_0 \in [0, 1]$ with $\varphi(t_0) = 0$. Put

$$A = t_0\tilde{A}_0 + (1-t_0)\tilde{A}_1,$$

then A is symmetric, $A \in A^I$ and $f(A) = 0$; hence, the assertion (viii) of Theorem 1 gives that A is singular, which is a contradiction. □

The necessary and sufficient condition of Theorem 3 requires only one matrix to be tested for positive definiteness. It bears a striking similarity with the characterization of nonnegative invertibility of interval matrices given in [11], Theorem 1 (each $A \in A^I$ is nonnegative invertible if and only if A^I is regular and $(A_c + \Delta)^{-1} \geqq 0$). However, the result is not as pleasant as it might seem because verifying regularity of an interval matrix is generally a difficult problem as it can be clearly seen from Theorem 5.1 in [12], where a number of necessary and sufficient regularity conditions are given, all of which require computation of at least 2^{n-1} quantities of some sort (as evaluating determinants, solving systems of linear equations, inverting matrices, and so on). Nevertheless, there exists an easily verifiable sufficient regularity condition that, in this author's experience, covers most practical examples. Employing it in Theorem 3 leads to this sufficient condition.

THEOREM 4. *Let $A^I = [A_c - \Delta, A_c + \Delta]$ be a symmetric interval matrix such that A_c is positive definite and*

$$\rho(|A_c^{-1}|\Delta) < 1 \tag{2}$$

holds. Then A^I is positive definite.

Proof. Because A_c is positive definite, it is invertible and the condition (2) guarantees regularity of A^I (see Beeck [2]). Hence, Theorem 3 gives that A^I is positive definite. □

We also note that if $(|A_c^{-1}|\Delta)_{jj} \geqq 1$ for some j, then A^I contains a singular matrix (assertion (iii) of Corollary 5.1 in [12]); hence, A^I is not positive definite.

Another sufficient condition can be derived from Theorem 1.

THEOREM 5. *Let a symmetric interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ satisfy*

$$\rho(\Delta) \leqq \lambda_{\min}(A_c). \tag{3}$$

Then A^I is positive semidefinite. Moreover, if the inequality (3) *holds sharply, then A^I is positive definite.*

Proof. According to the assertions (vii) and (ii) of Theorem 1, we have $\min\{f(A); A \in A^I\} \geqq \lambda_{\min}(A_c) - \rho(\Delta) \geqq 0$; hence, $f(A) \geqq 0$ for each $A \in A^I$, so that A^I is positive semidefinite. If (3) holds sharply, then $f(A) > 0$ for each $A \in A^I$; hence, A^I is positive definite. □

In the next section we shall apply the results obtained to characterize stability of symmetric interval matrices.

3. Hurwitz stability of interval matrices. A square matrix A is called Hurwitz stable (for the sake of brevity, we shall say only "stable") if $\mathrm{Re}\,\lambda < 0$ for each eigenvalue λ of A (in other words, if all its eigenvalues lie in the open left half of the complex plane). An interval matrix A^I is said to be stable if each $A \in A^I$ is stable. The problem of stability of interval matrices arises naturally in control theory in connection with the behavior of a linear time invariant system $\dot{x}(t) = Ax(t)$ under data perturbations and has been extensively studied recently; we refer the reader to the survey paper by Mansour [8] for a detailed list of references. We investigate here mainly stability of symmetric interval matrices, which turns out to be closely connected to the contents of the previous section due to the well-known result that states a symmetric matrix A is stable if and only if $-A$ is positive definite (see, e.g., [5]). However, some care must be taken because a symmetric interval matrix can contain nonsymmetric matrices whose eigenvalues are not real. As an example, consider the symmetric interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ with $A_c = 0$ and

$$\Delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which contains the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

whose eigenvalues are $\pm i$.

In contrast to the previous section where we employed the matrices $A_z = A_c - T_z\Delta T_z$, here we shall characterize stability in terms of matrices

$$\bar{A}_z = A_c + T_z\Delta T_z, \qquad z \in Y.$$

Obviously, $\bar{A}_z \in A^I$ and all $\bar{A}_z$ are symmetric if A^I is symmetric.

THEOREM 6. *Let $A^I = [A_c - \Delta, A_c + \Delta]$ be a symmetric interval matrix. Then the following assertions are equivalent:*

(a) *A^I is stable,*

(b) *$[-A_c - \Delta, -A_c + \Delta]$ is positive definite,*

(c) *$\bar{A}_z$ is stable for each $z \in Y$.*

Proof. We shall prove that $(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$. Let us denote $A_0^I = [-A_c - \Delta, -A_c + \Delta]$; notice that $A_0^I = \{-A; A \in A^I\}$.

$(a) \Rightarrow (c)$: The proof is obvious because $\bar{A}_z \in A^I$ for each $z \in Y$.

$(c) \Rightarrow (b)$: Let $z \in Y$. Because $\bar{A}_z$ is symmetric and stable, it follows that all its eigenvalues are negative; hence, the symmetric matrix

$$-\bar{A}_z = -A_c - T_z\Delta T_z$$

has all eigenvalues positive, so that it is positive definite [5]. But $-\bar{A}_z$ is just the matrix A_z for the interval matrix A_0^I; hence, A_0^I is positive definite by the assertion (c) of Theorem 2.

$(b) \Rightarrow (a)$: Let A_0^I be positive definite. Consider an eigenvalue λ of a matrix $A \in A^I$. Due to the Bendixson theorem ([15], p. 395), we have

$$\mathrm{Re}\,\lambda \leqq \lambda_{\max}\left(\tfrac{1}{2}(A + A^T)\right),$$

where the matrix $\tilde{A} = \frac{1}{2}(A + A^T)$ is symmetric and belongs to A^I; hence, $-\tilde{A} \in A_0^I$. Thus, $-\tilde{A}$ is positive definite so that all eigenvalues of $\tilde{A}$ are negative, which gives that $\mathrm{Re}\,\lambda \leqq \lambda_{\max}(\tilde{A}) < 0$. Hence, A is stable, and because it was chosen arbitrarily, A^I is also stable. □

There are several previous results relevant to the equivalence $(a) \Leftrightarrow (c)$. First, let us recall that a matrix $A \in A^I$ is called a vertex matrix of A^I if for each $i, j \in \{1, \ldots, n\}$, either $A_{ij} = (A_c - \Delta)_{ij}$ or $A_{ij} = (A_c + \Delta)_{ij}$ holds. Thus, there are exactly 2^{n^2} vertex matrices in the most disadvantageous case of $\Delta > 0$. Clearly, $\bar{A}_z$ is a vertex matrix for each $z \in Y$. The first attempt to use vertex matrices for characterizing stability was made by Białas [3], who proved that a general interval matrix A^I is stable if and only if all its vertex matrices are stable. His result was shown, however, to be erroneous by Karl, Greschak, and Verghese [7] and independently by Barmish and Hollot [1]. Soh [14] proved in 1990 that the conjecture is true for *symmetric* interval matrices in this form: if all the symmetric vertex matrices of A^I are stable, then each symmetric $A \in A^I$ is stable. This result required testing $2^{n(n+1)/2}$ vertex matrices for stability. This bound has been essentially improved recently by Hertz [6], who proved (using another notation) that if all the matrices $\bar{A}_z$ are stable, then each *symmetric* $A \in A^I$ is stable; this reduced the number of test matrices from $2^{n(n+1)/2}$ to 2^{n-1}. Theorem 6 shows that under the Hertz assumption *each* matrix $A \in A^I$ is already stable.

In Theorem 2 we showed that positive (semi)definiteness of a general interval matrix can be equivalently formulated in terms of the associated symmetric interval matrix A_s^I. Unfortunately, this nice property does not hold for stability, where only one implication is true.

THEOREM 7. *If A_s^I is stable, then A^I is also stable.*

Proof. Let λ be an eigenvalue of a matrix $A \in A^I$. Then by the Bendixson theorem we have $\operatorname{Re} \lambda \leq \lambda_{\max}(\frac{1}{2}(A + A^T)) < 0$ because the symmetric matrix $\frac{1}{2}(A + A^T)$ belongs to A_s^I and thus has all eigenvalues negative. This proves that A^I is stable. □

The converse implication is generally not valid. Consider the interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ with

$$A_c = \begin{pmatrix} 1 & 7 \\ -1 & -2 \end{pmatrix}$$

and $\Delta = 0$. Here A^I is stable because A_c is stable ($\operatorname{Re} \lambda_1 = \operatorname{Re} \lambda_2 = -\frac{1}{2}$), but A_s^I is not because $\lambda_{\max}(A_c') = (\sqrt{45} - 1)/2 = 2.85\cdots$.

Finally, we give the respective versions of Theorems 3 and 4 for the case of stability. The reformulations are direct consequences of the equivalence $(a) \Leftrightarrow (b)$ of Theorem 6.

THEOREM 8. *A symmetric interval matrix A^I is stable if and only if it is regular and contains at least one stable symmetric matrix.*

Proof. The "only if" part follows from the fact that each stable matrix is nonsingular. Conversely, if A^I is regular and contains a stable symmetric matrix $\tilde{A}$, then $A_0^I = [-A_c - \Delta, -A_c + \Delta] = \{-A;\ A \in A^I\}$ is also regular and contains a positive definite matrix $-\tilde{A}$; hence, A_0^I is positive definite by Theorem 3 and A^I is stable by Theorem 6. □

The last result of this section follows from Theorem 4 applied to the interval matrix $[-A_c - \Delta, -A_c + \Delta]$ and its straightforward proof is omitted.

THEOREM 9. *Let $A^I = [A_c - \Delta, A_c + \Delta]$ be a symmetric interval matrix such that A_c is stable and*

$$\rho(|A_c^{-1}|\Delta) < 1$$

holds. Then A^I is stable.

For a practical verification, the results of this section can be used in the following way. Given an interval matrix A^I, first form the symmetric interval matrix A_s^I and test it for stability using Theorem 9. If the test is successful, then A^I is stable (Theorem 7).

This procedure will, however, fail whenever A^I is stable, whereas A_s^I is not, as, e.g., in the example following Theorem 7. In such a case another condition must be tried (cf. Mansour [8] for further results).

Example. Consider the interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ with

$$A_c = \begin{pmatrix} -1 & -1 & 2 \\ 3 & -2 & -5 \\ -2 & 1 & -5 \end{pmatrix}$$

and $\Delta_{ij} = 0.03$ for each i, j. Then for the associated symmetric interval matrix $A_s^I = [A_c' - \Delta', A_c' + \Delta']$, we have

$$A_c' = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & -2 \\ 0 & -2 & -5 \end{pmatrix}$$

and $\Delta' = \Delta$. Because A_c' is stable and $\rho(|(A_c')^{-1}|\Delta') = 0.9 < 1$, Theorems 7 and 9 imply that A^I is stable.

4. Schur stability of interval matrices. A square matrix A is called Schur stable if $\rho(A) < 1$, i.e., if $|\lambda| < 1$ for each eigenvalue λ of A. We shall consider here Schur stability of symmetric matrices only to avoid complex eigenvalues that seemingly cannot be easily handled by the method used. Therefore, we shall say that a symmetric interval matrix A^I is Schur stable if each *symmetric* matrix $A \in A^I$ is Schur stable; hence, we do not take into account the nonsymmetric matrices contained in A^I. This definition is in accordance with the approach employed in [14] or [6].

A necessary and sufficient condition for Schur stability has been recently given by Hertz [6], who proved that a symmetric interval matrix A^I is Schur stable if and only if all the matrices $A_z, \bar{A}_z, z \in Y$ are Schur stable. In Theorem 11 below we formulate another necessary and sufficient condition based on the following result that links Schur stability to Hurwitz stability.

THEOREM 10. *A symmetric interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ is Schur stable if and only if the symmetric interval matrices*

$$[(A_c - I) - \Delta, (A_c - I) + \Delta] \tag{4}$$

and

$$[(-A_c - I) - \Delta, (-A_c - I) + \Delta] \tag{5}$$

are stable, where I is the unit matrix.

Proof. Only if: Denote the interval matrix (4) by $\tilde{A}^I$ and let $\tilde{A}_z = (A_c - I) + T_z\Delta T_z = \bar{A}_z - I$ for $z \in Y$. Because $\bar{A}_z$ is symmetric and Schur stable, it has all eigenvalues in $(-1, 1)$; therefore, all the eigenvalues of $\tilde{A}_z$ belong to $(-2, 0)$; hence, $\tilde{A}_z$ is stable. In view of Theorem 6 this implies that $\tilde{A}^I$ is stable. Stability of (5) can be proved in a similar way if we consider the matrices $\tilde{\tilde{A}}_z = -A_z - I$, $z \in Y$.

If: Let $A \in A^I$ be symmetric and let λ be an eigenvalue of A. Then $\lambda - 1$ is an eigenvalue of the matrix $A - I$ that belongs to (4) and hence is stable, which gives $\lambda - 1 < 0$. In a similar way, stability of (5) implies $-\lambda - 1 < 0$. Hence, $|\lambda| < 1$; thus, A^I is Schur stable. □

Now we have this criterion that is again formulated along the lines of Theorems 3 and 8.

THEOREM 11. *A symmetric interval matrix A^I is Schur stable if and only if it contains at least one Schur stable symmetric matrix and both the interval matrices* (4) *and* (5) *are regular.*

Proof. Only if: If A^I is Schur stable, then both (4) and (5) are stable by Theorem 10; hence, regular. If: Let some symmetric $A_0 \in A^I$ be Schur stable and let (4) and (5) be regular. Then $A_0 - I$ is symmetric, stable, and belongs to (4); hence, (4) is stable by Theorem 8. Similarly, stability of (5) can be established by considering the matrix $-A_0 - I$. Then Theorem 10 gives that A^I is Schur stable. □

Again, using sufficient regularity condition, we obtain the following.

THEOREM 12. *Let $A^I = [A_c - \Delta, A_c + \Delta]$ be a symmetric interval matrix such that A_c is Schur stable and the conditions*

$$\rho(|A_c - I|^{-1}\Delta) < 1 \tag{6}$$

$$\rho(|A_c + I|^{-1}\Delta) < 1 \tag{7}$$

are satisfied. Then A^I is Schur stable.

Proof. This is a direct consequence of Theorem 11 because (6) and (7) are the Beeck sufficient regularity conditions [2] for the interval matrices (4) and (5). □

5. Final remark. In Theorems 4, 9, and 12 we formulated verifiable sufficient conditions for positive definiteness, Hurwitz and Schur stability of symmetric interval matrices. Each of them involved the sufficient condition (2) for regularity of an interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$. This condition may be seen to be inappropriate for practical computations because the inverse matrix computed on a computer is usually afflicted with roundoff errors. Therefore, for practical purposes we propose a modified condition

$$\rho(|I - QA_c| + |Q|\Delta) < 1 \tag{8}$$

involving an arbitrary square matrix Q, because we have: *if (8) holds for some Q, then A^I is regular*. In fact, for an arbitrary $A \in A^I$, we have

$$QA = I - (I - QA_c + Q(A_c - A))$$

and because

$$\rho(I - QA_c + Q(A_c - A)) \leqq \rho(|I - QA_c| + |Q|\Delta) < 1,$$

it follows that QA is nonsingular; hence, A is nonsingular. Notice that (2) is a special case of (8) for $Q = A_c^{-1}$. In practical computations we recommend to set Q equal to the *computed* value of A_c^{-1}.

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