ZAMM · Z. angew. Math. Mech. 74 (1994) 6, T 688 Akademie Verlag

ROHN, J.

On Some Properties of Interval Matrices Preserved by Nonsingularity

MSC (1991): 65G10, 15A48, 15A09

In this note we show that for several frequently used properties of square matrices there holds: *each matrix in an interval matrix A^I has a property* P *if and only if A^I is nonsingular and contains at least one matrix having the property* P.

We begin with listing the properties in question. A square matrix $A = (a_{ij})$ is called *inverse nonnegative* if $A^{-1} \geqq 0$, and *inverse positive* if $A^{-1} > 0$ (componentwise inequalities); it is called an M-*matrix* if $A^{-1} \geqq 0$ and $a_{ij} \leqq 0$ for $i \neq j$. More generally, for two signature matrices S_1, S_2 (i.e., diagonal matrices with ± 1 diagonal entries) we say that A is *of inverse sign pattern* (S_1, S_2) if $S_1 A^{-1} S_2 \geqq 0$, and that A is *of strict inverse sign pattern* (S_1, S_2) if $S_1 A^{-1} S_2 > 0$. A matrix A (not necessarily symmetric) is called *positive definite* if $x^T A x > 0$ for each $x \neq 0$, and *(Hurwitz) stable* if $\operatorname{Re} \lambda < 0$ for each eigenvalue λ of A.

An interval matrix is a set of matrices $A^I = [\underline{A}, \bar{A}] = \{A; \underline{A} \leqq A \leqq \bar{A}\}$, where $\underline{A}$ and $\bar{A}$ are square, $\underline{A} \leqq \bar{A}$. We say that A^I *has a property* P (inverse nonnegative, ..., stable, nonsingular) if each $A \in A^I$ has the property P. The following theorem shows that all the above-listed properties are preserved by nonsingularity:

Theorem: *Let A^I be nonsingular. Then we have:*
(i) *A^I is inverse nonnegative if and only if $\bar{A}^{-1} \geqq 0$,*
(ii) *A^I is inverse positive if and only if $\bar{A}^{-1} > 0$,*
(iii) *A^I is an* M-*matrix if and only if $\bar{A}$ is an* M-*matrix,*
(iv) *A^I is of inverse sign pattern (S_1, S_2) if and only if the matrix $\frac{1}{2}(\underline{A} + \bar{A}) + \frac{1}{2} S_2(\bar{A} - \underline{A}) S_1$ is of inverse sign pattern (S_1, S_2),*
(v) *A^I is of strict inverse sign pattern (S_1, S_2) if and only if the matrix $\frac{1}{2}(\underline{A} + \bar{A}) + \frac{1}{2} S_2(\bar{A} - \underline{A}) S_1$ is of strict inverse sign pattern (S_1, S_2).*

Moreover, if both $\underline{A}$ and $\bar{A}$ are symmetric, then
(vi) *A^I is positive definite if and only if at least one $A \in A^I$ is positive definite,*
(vii) *A^I is stable if and only if at least one symmetric $A \in A^I$ is stable.*

Proof: The "only if" parts follow obviously from the definition; thus we are confined to prove the "if" ones. (i) is proved in [3], Theorem 1, assertion (iv). (ii): From (i) we have that A^I is inverse nonnegative; for each $A \in A^I$, premultiplying the inequality $A \leqq \bar{A}$ by A^{-1} and $\bar{A}^{-1}$ we obtain $A^{-1} \geqq \bar{A}^{-1} > 0$, hence A^I is inverse positive. (iii): A^I is inverse nonnegative by (i) and for each $A \in A^I$ we have $A_{ij} \leqq \bar{A}_{ij} \leqq 0$ for $i \neq j$, implying that A^I is an M-matrix. (iv): An obvious computation shows that $\{S_2 A S_1; A \in A^I\} = [S_2 \underset{\sim}{A} S_1, S_2 \tilde{A} S_1]$, where $\underset{\sim}{A} = \frac{1}{2}(\underline{A} + \bar{A}) - \frac{1}{2} S_2(\bar{A} - \underline{A}) S_1$ and $\tilde{A} = \frac{1}{2}(\underline{A} + \bar{A}) + \frac{1}{2} S_2(\bar{A} - \underline{A}) S_1$. Since $(S_2 \tilde{A} S_1)^{-1} = S_1 \tilde{A}^{-1} S_2 \geqq 0$ by assumption, we have in the light of (i) that $[S_2 \underset{\sim}{A} S_1, S_2 \tilde{A} S_1]$ is inverse nonnegative, hence for each $A \in A^I$ we obtain $S_1 A^{-1} S_2 = (S_2 A S_1)^{-1} \geqq 0$. (v) follows in a similar way from (ii). These assertions (vi) and (vii) are proved in [4], Theorems 3 and 8. ■

Let us note that the assertions (i)–(v) are rather of theoretical interest since there exists a verifiable necessary and sufficient inverse nonnegativity condition due to KUTTLER [1] which can be easily extended to the cases (ii)–(v) as well. The cases (vi) and (vii) are different since no such a condition is known for them. Here, employing the sufficient nonsingularity condition

$$\varrho(|\underline{A} + \bar{A}|^{-1} (\bar{A} - \underline{A})) < 1$$

(cf. [2]), we obtain from (vi), (vii) verifiable sufficient conditions for positive definiteness or stability of A^I.

References

1 KUTTLER, J.: A fourth-order finite-difference approximation for the fixed membrane eigenproblem. Math. Computat. **25** (1971), 237–256.
2 NEUMAIER, A.: Interval methods for systems of equations. Cambridge University Press, Cambridge 1990.
3 ROHN, J.: Inverse-positive interval matrices. Z. Angew. Math. Mech. **67** (1987), T 492–T 493.
4 ROHN, J.: Positive definiteness and stability of interval matrices. SIAM J. Matrix Anal. Appl. **15** (1994), 175–184.

Address: Dr. JIŘÍ ROHN, Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, CZ-11800 Prague, Czech Republic