LINEAR INTERVAL INEQUALITIES

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Abstract

We prove that a system of linear inequalities with interval-valued data is weakly solvable (each system obtained by fixing coefficients in the intervals prescribed has a solution) if and only if it is strongly solvable (all such systems have a solution in common) and describe an algorithm for checking strong solvability.

1 Introduction

In this paper we study systems of linear interval inequalities

$$A^I x \leq b^I \tag{1}$$

where $A^I = \{A; \underline{A} \leq A \leq \overline{A}\}$ (componentwise inequalities) is an $m \times n$ interval matrix and $b^I = \{b; \underline{b} \leq b \leq \overline{b}\}$ is an m-dimensional interval vector. Under the formally written system (1) we understand the family of systems of linear inequalities

$$Ax \leq b \tag{2}$$

for all A and b satisfying

$$A \in A^I, \quad b \in b^I. \tag{3}$$

We introduce two concepts of solvability:

1. a system (1) is called *weakly solvable* if for each $A \in A^I$, $b \in b^I$ the system (2) has a solution (which generally depends on A and b)

2. a system (1) is called *strongly solvable* if there exists an x_0 satisfying $Ax_0 \leq b$ for each $A \in A^I$, $b \in b^I$ (i.e., if all the systems (2), (3) have a solution in common).

We prove in Theorem 1 below that, rather surprisingly, weak and strong solvability are equivalent, and describe a method for verifying strong (equivalently, weak) solvability of (1) by means of solving only one system of linear (not interval) inequalities. As a by-product of the proof, we obtain that if (1) is not strongly solvable, then it contains an unsolvable system $A_0 x \leq \underline{b}$ with a matrix $A_0 \in A^I$ of a very special form: for each j, all but at most one of the coefficients of the j-th column of A_0 are fixed either at the lower, or at the upper bound of the respective interval (Theorem 2), and we describe a method for constructing such a matrix A_0 via solving an auxiliary linear programming problem. The results obtained are summed up in the form of an algorithm. Several relevant results are mentioned in the concluding remarks.

2 The results

To facilitate formulations, let us call a vector x_0 a *strong solution* of (1) if

$$Ax_0 \leq b$$

holds for each $A \in A^I$, $b \in b^I$; hence, the above-defined strong solvability is equivalent to the existence of a strong solution to (1). Let us denote the set of all strong solutions of (1) by X_S. We have this description:

Proposition. *For a system (1) we have*

$$X_S = \left\{ x_1 - x_2; \overline{A}x_1 - \underline{A}x_2 \leq \underline{b}, x_1 \geq 0, x_2 \geq 0 \right\}.$$

Proof . Let $x \in X_S$. Put $x_1 = x^+ = max\{x, 0\}$ (componentwise) and $x_2 = x^- = max\{-x, 0\}$. Then $x_1 \geq 0$, $x_2 \geq 0$ and $x = x_1 - x_2$. Furthermore define a matrix A columnwise by $A_{.j} = \overline{A}_{.j}$ if $x_j \geq 0$ and $A_{.j} = \underline{A}_{.j}$ if $x_j < 0$, so that $A \in A^I$. Then, since x is a strong solution, we have $\overline{A}x_1 - \underline{A}x_2 = Ax \leq \underline{b}$. Conversely, let x_1, x_2 be a nonnegative solution to $\overline{A}x_1 - \underline{A}x_2 \leq \underline{b}$ and let $x = x_1 - x_2$. Then for each $A \in A^I$, $b \in b^I$ we have that $Ax = A(x_1 - x_2) \leq \overline{A}x_1 - \underline{A}x_2 \leq \underline{b} \leq b$, hence x is a strong solution. ■

Now we have this main result:

Theorem 1 *A system of linear interval inequalities (1) is weakly solvable if and only if it is strongly solvable.*

We shall prove Theorem 1 together with this result:

Theorem 2 *Let a system (1) be not strongly solvable. Then it contains an unsolvable system*

$$A_0 x \leq \underline{b}$$

where A_0 is of the following form: for each $j \in \{1, \dots, n\}$ there is an $i_j \in \{1, \dots, m\}$ such that

$$(A_0)_{ij} \begin{cases} = \underline{A}_{ij} & \textit{for} \quad i < i_j, \\ \in \left[\underline{A}_{ij}, \overline{A}_{ij}\right] & \textit{for} \quad i = i_j, \\ = \overline{A}_{ij} & \textit{for} \quad i > i_j. \end{cases} \tag{4}$$

Proof of Theorems 1 and 2. Obviously, a strongly solvable system (1) is also weakly solvable. To prove Theorem 2 and also the "only if" part of Theorem 1, assume that (1) is not strongly solvable. Then, according to the Proposition, the system of linear inequalities

$$\begin{gathered} \overline{A}x_1 - \underline{A}x_2 \leq \underline{b} \\ x_1 \geq 0, \quad x_2 \geq 0 \end{gathered}$$

does not have a solution, which in view of Farkas lemma implies the existence of a vector $y \geq 0$ satisfying $\overline{A}^T y \geq 0$, $\underline{A}^T y \leq 0$ and $\underline{b}^T y < 0$. For each $j \in \{1, \dots, n\}$ and $k \in \{0, \dots, m\}$ define a number t_k^j by

$$t_k^j = \sum_{i \leq k} y_i \underline{A}_{ij} + \sum_{i > k} y_i \overline{A}_{ij}$$

(employing a usual convention that $\sum_\emptyset = 0$). Then we have

$$\prod_{k=1}^{m} t_{k-1}^j t_k^j = \left(\prod_{k=1}^{m-1} t_k^j\right)^2 t_0^j t_m^j = \left(\prod_{k=1}^{m-1} t_k^j\right)^2 \left(\overline{A}^T y\right)_j \left(\underline{A}^T y\right)_j \le 0,$$

hence there exists a $k \in \{1, \ldots, m\}$ satisfying

$$t_{k-1}^j t_k^j \le 0.$$

But since

$$t_{k-1}^j t_k^j = \left(\sum_{i<k} y_i \underline{A}_{ij} + y_k \overline{A}_{kj} + \sum_{i>k} y_i \overline{A}_{ij}\right) \left(\sum_{i<k} y_i \underline{A}_{ij} + y_k \underline{A}_{kj} + \sum_{i>k} y_i \overline{A}_{ij}\right),$$

we see that there exists an $\alpha^j \in [\underline{A}_{kj}, \overline{A}_{kj}]$ such that

$$\sum_{i<k} y_i \underline{A}_{ij} + y_k \alpha^j + \sum_{i>k} y_i \overline{A}_{ij} = 0 \tag{5}$$

holds. Now put $i_j = k$ and define A_0 by

$$(A_0)_{ij} = \begin{cases} \underline{A}_{ij} & \text{if} \quad i < i_j, \\ \alpha^j & \text{if} \quad i = i_j, \\ \overline{A}_{ij} & \text{if} \quad i > i_j \end{cases}$$

($i = 1, \ldots, m$, $j = 1, \ldots, n$), then A_0 is of the form (4), $A_0 \in A^I$ and from (5) we have $A_0^T y = 0$ which together with $y \ge 0$, $\underline{b}^T y < 0$ implies (again using Farkas lemma) that the system of linear inequalities

$$A_0 x \le \underline{b}$$

does not have a solution. This proves Theorem 2, and by contradiction also the "only if" part of Theorem 1. ∎

Let us notice that the proof of Theorem 2 gives a method for constructing a matrix A_0 provided a nonnegative vector y satisfying $\overline{A}^T y \ge 0$, $\underline{A}^T y \le 0$, $\underline{b}^T y < 0$ is known. Such a vector can be found by solving the linear programming problem

$$\min\left\{\underline{b}^T y; \overline{A}^T y \ge 0, \underline{A}^T y \le 0, 0 \le y \le e\right\} \tag{6}$$

where $e = (1, 1, \ldots, 1)^T$. In fact, the set of feasible solutions of (6) is nonempty ($y = 0$ is feasible) and bounded, hence (6) has a finite optimum. If (1) is not strongly solvable, then from the proof of Theorem 2 (where we can normalize y to achieve $y \leq e$) we know that the optimal value of (6) is negative. Hence any optimal solution y to (6) satisfies $\underline{b}^T y < 0$ and can be used for construction of a matrix A_0.

We can sum up our results into the form of a simple algorithm:
Algorithm (for checking solvability of (1)).

1. Solve the system of linear inequalities

$$\begin{aligned} \overline{A}x_1 - \underline{A}x_2 \leq \underline{b} \\ x_1 \geq 0, \quad x_2 \geq 0 \end{aligned} \tag{7}$$

   by any known method (e.g. by phase I of the simplex algorithm).

2. If a solution x_1, x_2 to (7) is found, set $x = x_1 - x_2$ and terminate: (1) is strongly solvable and x is a strong solution to it.

3. If (7) does not have a solution, find an optimal solution y of the linear program (6).

4. Using y, construct A_0 as in the proof of Theorem 2 and terminate: (1) is not weakly (nor strongly) solvable and the system $A_0 x \leq \underline{b}$ does not have a solution.

3 Concluding remarks

We conclude with mentioning several related results.

1. The description of the set of strong solutions X_S in the Proposition is similar to that one used in [2] for characterizing the set of "tolerance solutions" of a system of linear interval equations.

2. A result related to the form of the matrix A_0 in Theorem 2 can be found in [3], Theorem 5.1, assertion (C7), where it is proved that if a

square interval matrix A^I contains a singular matrix at all, then it also contains a singular matrix A_0 of the form

$$(A_0)_{ij} \begin{cases} \in \left\{\underline{A}_{ij}, \overline{A}_{ij}\right\} & \text{for} \quad (i,j) \neq (k,m) \\ \in \left[\underline{A}_{i,j}, \overline{A}_{ij}\right] & \text{for} \quad (i,j) = (k,m) \end{cases}$$

for some pair of indices (k, m).

3. It follows from the result by Gerlach [1] that a vector x satisfies $Ax \leq b$ for *some* $A \in A^I$, $b \in b^I$ if and only if it solves a system

$$\left(\frac{1}{2}(\underline{A} + \overline{A}) - \frac{1}{2}(\overline{A} - \underline{A})S\right) x \leq \bar{b}$$

for some signature matrix S (i.e., a diagonal matrix with diagonal entries 1 or -1). Since there are altogether 2^n signature matrices, it appears that the problem of verifying that *no* system $Ax \leq b$ with data satisfying $A \in A^I$, $b \in b^I$ has a solution is much more difficult than that of verifying strong solvability.

References

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