COMPUTING EXACT COMPONENTWISE BOUNDS ON SOLUTIONS OF LINEAR SYSTEMS WITH INTERVAL DATA IS NP-HARD*

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Abstract. We prove that it is NP-hard to compute the exact componentwise bounds on solutions of all the linear systems which can be obtained from a given linear system with a nonsingular matrix by perturbing all the data independently of each other within prescribed tolerances.

Key words. linear equations, perturbation, componentwise bounds

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1. Introduction.

Given a system of linear equations

$$Ax = b \tag{1}$$

where $A \in R^{n\times n}$ is nonsingular and $b \in R^n$, consider the perturbed system

$$A'x' = b' \tag{2}$$

with data A', b' satisfying

$$|A' - A| \le \Delta \tag{3}$$

and

$$|b' - b| \le \delta \tag{4}$$

where $\Delta \in R_+^{n\times n}$ and $\delta \in R_+^n$ are correspondingly the matrix and vector of perturbation bounds (the absolute value of a matrix $B = (b_{ij})$ is defined by $|B| = (|b_{ij}|)$, and the inequality (3) is understood componentwise; similarly for vectors). Let X denote the set of solutions of all the perturbed systems, i.e.

$$X = \{x'; A'x' = b' \text{ for some } A', b' \text{ satisfying } (3), (4)\}.$$

Naturally, we are interested in knowing the exact range of the components of the solution under the allowed perturbations, i.e. in computing the numbers

$$\underline{x}_i = \min_{x'\in X} x'_i \tag{5}$$

$$\overline{x}_i = \max_{x'\in X} x'_i \tag{6}$$

$(i = 1, \ldots, n)$; we call them the *exact componentwise bounds* on solutions of the perturbed systems.

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During the last almost 30 years, the problem of computing the exact componentwise bounds (formulated often in the framework of systems of linear interval equations) has received much attention. General methods (assuming only nonsingularity of each matrix A' satisfying (3)) were given by Oettli [7], Rohn [9] and Shary [12]; however, all of them require in the worst case an amount of operations which is exponential in n. As a result, these methods are not applicable to problems of large dimension n. Therefore, a number of articles deal with special cases (such as M-matrices [2], H-matrices [6], inverse stable matrices [9], matrices satisfying a spectral condition [11] or diagonally dominant [4], etc.) for which there exist polynomial algorithms for computing the exact componentwise bounds (or their enclosures). For surveys of such methods, see the monographes by Alefeld and Herzberger [1] or Neumaier [6].

In this paper we show that computing the exact componentwise bounds is NP-hard (see Garey and Johnson [3] for basic concepts of the complexity theory). Thus, unless $P = NP$ (which is currently widely believed not to be true), we cannot expect an existence of polynomial-time algorithms for solving our problem. The NP-hardness of the computation of (5), (6) for overdetermined systems (A of size $m \times n$, $m > n$) was recently established by Kreinovich et al. [5], but the idea of the proof, which reduces 3-satisfiability to computation of the exact componentwise bounds for linear systems with matrices of size about $3n \times n$, cannot be used for the square case.

We carry out the proof of our result by studying a special instance of *constant* componentwise perturbations. We show that in this case the optimal value of a specially chosen linear function over X can be expressed in terms of the reciprocal value of the so-called radius of nonsingularity which has been recently shown to be NP-hard to compute (Poljak and Rohn [8]). Then adding one more row and column to the original system to make the linear function depend on a single variable only, we obtain the desired result.

2. Auxiliary results. For a given nonsingular matrix $A \in R^{n\times n}$ and the linear system

$$Ax = 0$$

(which has a unique solution $x = 0$), consider the perturbed systems

$$A'x' = b'$$

with

$$|A' - A| \leq \beta ee^T \tag{7}$$

and

$$|b'| \leq \beta e, \tag{8}$$

where $e = (1, 1, \dots, 1)^T \in R^n$ and β is a real parameter. To underline the dependence on the parameter, let us denote the solution set by X_β:

$$X_\beta = \{x'; A'x' = b' \text{ for some } A', b' \text{ satisfying } (7), (8)\}.$$

We shall first give a description of the set X_β; throughout the following text, we use the norm $\|x\| = \|x\|_1 = e^T|x| = \sum_i |x_i|$.

PROPOSITION 2.1. *Let A be nonsingular, and let β satisfy*

$$0 < \beta < \frac{1}{e^T|A^{-1}|e}. \tag{9}$$

Then each A' satisfying (7) is nonsingular and we have

$$X_\beta = \{x'; x' = \frac{\beta}{1-\beta\|A^{-1}c\|}A^{-1}c, -e \le c \le e\}. \tag{10}$$

Proof.
1) Let $x' \in X_\beta$, i.e. $A'x' = b'$ for some A', b' satisfying (7), (8). Then we have $|Ax'| = |(A - A')x' + b'| \le \beta ee^T|x'| + \beta e = \beta(\|x'\| + 1)e$, hence if we take

$$c = \frac{1}{\beta(\|x'\| + 1)}Ax',$$

then we have $-e \le c \le e$ and $Ax' = \beta(\|x'\| + 1)c$, which implies

$$x' = \beta(\|x'\| + 1)A^{-1}c, \tag{11}$$

hence

$$\|x'\| = \beta(\|x'\| + 1)\|A^{-1}c\|. \tag{12}$$

Since

$$\beta\|A^{-1}c\| = \beta e^T|A^{-1}c| \le \beta e^T|A^{-1}|e < 1 \tag{13}$$

due to (9), from (12) we obtain

$$\|x'\| = \frac{\beta\|A^{-1}c\|}{1-\beta\|A^{-1}c\|}.$$

Substituting this equality into (11) leads to

$$x' = \frac{\beta}{1-\beta\|A^{-1}c\|}A^{-1}c, \tag{14}$$

hence x' is of the form described in (10).
2) Conversely, let x' be of the form (14) for some c satisfying $-e \le c \le e$. Define a vector $z \in R^n$ as follows: $z_j = 1$ if $x'_j \ge 0$ and $z_j = -1$ otherwise ($j = 1, \dots, n$). Then $z^T x' = e^T|x'| = \|x'\|$, hence

$$(A - \beta cz^T)x' = \frac{1}{1-\beta\|A^{-1}c\|}(\beta c - \beta^2\|A^{-1}c\|c) = \beta c$$

which means that x' is a solution of the system

$$(A - \beta cz^T)x' = \beta c$$

where $|(A - \beta cz^T) - A| = \beta|c| \cdot |z|^T \le \beta ee^T$ and $|\beta c| \le \beta e$. Hence, $x' \in X_\beta$.
3) From (13) and (14), we conclude that

$$\|x'\| \le \frac{\beta e^T|A^{-1}|e}{1-\beta e^T|A^{-1}|e}$$

for each $x' \in X_\beta$, hence X_β is bounded. If some A' satisfying (7) was singular, then we would have $A'x' = 0$ for some $x' \neq 0$, hence $\lambda x' \in X_\beta$ for each $\lambda \in R^1$, which would contradict the boundedness of X_β. Hence, each A' that satisfies (7) is nonsingular. □

Before proceeding further, let us introduce, for a matrix $B \in R^{n\times n}$, the number

$$r(B) = \max\{\|By\|; y \in \{-1,1\}^n\}.$$

A simple reasoning shows that it can be also written as

$$r(B) = \max\{z^T By; z, y \in \{-1,1\}^n\}$$

which is the form in which it was originally introduced in [8]. Then, we have the following result:

Proposition 2.2. *Let A be nonsingular, and let β satisfy (9). Then for each $i \in \{1,\dots,n\}$ we have*

$$\max_{x'\in X_\beta} (Ax')_i = \frac{\beta}{1-\beta r(A^{-1})}. \tag{15}$$

Proof. 1) First, we shall prove that

$$\|A^{-1}c\| \leq r(A^{-1}) \tag{16}$$

holds for each c, $|c| \leq e$. For every c that satisfies this inequality $|c| \leq e$, let's define vectors $z, y \in \{-1,1\}^n$ as follows: $z_j = 1$ if $(A^{-1}c)_j \geq 0$ and $z_j = -1$ otherwise $(j = 1,\dots,n)$, and $y_j = 1$ if $(z^T A^{-1})_j \geq 0$ and $y_j = -1$ otherwise $(j = 1,\dots,n)$. Then, we have $\|A^{-1}c\| = e^T|A^{-1}c| = z^T A^{-1}c \leq z^T A^{-1} y \leq \max\{z^T A^{-1} y; z, y \in \{-1,1\}^n\} = r(A^{-1})$, i.e., (16).

2) Let us fix an $i \in \{1,\dots,n\}$ and let $x' \in X_\beta$. According to Proposition 1, we have

$$x' = \frac{\beta}{1-\beta\|A^{-1}c\|}A^{-1}c$$

for some c such that $|c| \leq e$. Since the denominator is positive (due to (9) and (13)), we have

$$(Ax')_i \leq |(Ax')_i| \leq \frac{\beta}{1-\beta\|A^{-1}c\|} \leq \frac{\beta}{1-\beta r(A^{-1})}$$

(due to (16)). Hence,

$$\max_{x'\in X_\beta} (Ax')_i \leq \frac{\beta}{1-\beta r(A^{-1})}. \tag{17}$$

3) Take $\overline{y} \in \{-1,1\}^n$ such that

$$\|A^{-1}\overline{y}\| = \max\{\|A^{-1}y\|; y \in \{-1,1\}^n\} = r(A^{-1}).$$

Since $\|A^{-1}(-y)\| = \|A^{-1}y\|$, $\overline{y}$ can be chosen in such a way that $\overline{y}_i = 1$. According to Proposition 1, the vector

$$x' = \frac{\beta}{1-\beta\|A^{-1}\overline{y}\|}A^{-1}\overline{y}$$

belongs to X_β and satisfies the equality

$$(Ax')_i = \frac{\beta}{1-\beta r(A^{-1})},$$

hence the upper bound in (17) is achieved, which proves (15). ∎

3. NP-hardness. Now we are able to prove the main result.

THEOREM 3.1. *For an instance n, A, b, Δ, δ and $i \in \{1, \ldots, n\}$ such that each matrix A' satisfying (3) is nonsingular, computing both $\underline{x}_i$ and $\overline{x}_i$ given by (5) and (6) is NP-hard.*

Comment. Since checking nonsingularity of all matrices A' satisfying (3) is already NP-hard [8], we must include nonsingularity into the assumptions to separate the two problems.

Proof. In [8], Theorem 2.6 it is proved that computing $r(B)$ is NP-hard for $B \in R^{n\times n}$. The result was stated there for general matrices, but it remains valid if we confine ourselves to nonsingular matrices only (since the proof employs a diagonally dominant matrix, which is nonsingular). We will show that computing $r(B)$ can be polynomially reduced to the computation of an exact componentwise bound.

For a given nonsingular $B \in R^{n\times n}$, choose a β satisfying

$$0 < \beta < \frac{1}{e^T|B|e} \tag{18}$$

and compute $A = B^{-1}$ (this can be done in polynomial time). Now, construct the $(n+1)\times(n+1)$ matrices

$$\tilde{A} = \begin{pmatrix} A & 0 \\ A_{n\cdot} & -1 \end{pmatrix}$$

where $A_{n\cdot}$ denotes the nth row of A, and

$$\Delta = \begin{pmatrix} \beta e e^T & 0 \\ 0 & 0 \end{pmatrix}$$

and let

$$b = 0$$

and

$$\delta = \begin{pmatrix} \beta e \\ 0 \end{pmatrix}$$

($e \in R^n$). Then each $A' \in R^{(n+1)\times(n+1)}$ with $|A' - \tilde{A}| \le \Delta$ is nonsingular by (18) and by Proposition 1, and for the solution set of the perturbed systems we have

$$X = \{(x, x_{n+1})^T; x \in R^n, x \in X_\beta, x_{n+1} = A_{n\cdot}x\}.$$

Hence, for the exact componentwise bound on x_{n+1}, we conclude from Proposition 2 that

$$\overline{x}_{n+1} = \max_{x'\in X_\beta} (Ax')_n = \frac{\beta}{1-\beta r(B)}.$$

So, the computation of $r(B)$ has been polynomially reduced to the computation of $\overline{x}_{n+1}$. Thus, since computing $r(B)$ is NP-hard, the same must be true for $\overline{x}_{n+1}$ as well. In this way we have proved the NP-hardness of computing the exact upper bound on the highest index variable; now by permutation of variables we easily extend this result to an arbitrary variable. The statement for lower bounds follows immediately from the result just proved if we observe that the lower bounds differ only in their signs from the upper bounds for the system $Ax = -b$ under the same Δ and δ. ∎

Final note. The result can be made more understandable if we point out that (15) is, in general, a nonconvex optimization problem. Indeed, a lengthy argument (which we omit here) based on Theorems 1 and 2 in [10] proves that *if* $n \geq 3$, A *is nonsingular and* β *satisfies (9), then* X_β *is a nonconvex set whose convex hull has* 2^n *vertices which are exactly those points* x' *in (10) that correspond to parameter values* $t \in \{-1, 1\}^n$.

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