A perturbation theorem for linear equations

Jiří Rohn*
Faculty of Mathematics and Physics, Charles University,
Malostranské nám. 25, 118 00 Prague 1,
Czech Republic

We describe here explicit formulae for componentwise bounds on solution of a system of linear equations

$$Ax = b$$

(A square) under perturbation of all data. To make the result numerically tractable, we avoid the use of exact inverses, using instead some matrices R and M required only to satisfy certain inequalities. Hansen's optimality result [1], [2] is a special case of our theorem. Notations used: I is the unit matrix, ϱ denotes the spectral radius, for $A = (a_{ij})$ we denote $|A| = (|a_{ij}|)$ and inequalities are understood componentwise.

Theorem 1 *Let $A, \Delta \in R^{n\times n}$, $b, \delta \in R^n$, $\Delta \geq 0$, $\delta \geq 0$ and let R and M be arbitrary matrices satisfying*

$$\begin{aligned} MG + I &\leq M, \\ M &\geq 0, \end{aligned} \tag{1}$$

where

$$G = |I - RA| + |R|\Delta.$$

Then for each A' and b' such that

$$\begin{aligned} |A' - A| &\leq \Delta, \\ |b' - b| &\leq \delta, \end{aligned}$$

A' is nonsingular and the solution of the system

$$A'x' = b'$$

for each $i \in \{1, \ldots, n\}$ satisfies

$$\min\left\{\frac{\underset{\sim}{x}_i}{\alpha_i}, \frac{\underset{\sim}{x}_i}{\beta_i}\right\} \leq x'_i \leq \max\left\{\frac{\tilde{x}_i}{\alpha_i}, \frac{\tilde{x}_i}{\beta_i}\right\}, \tag{2}$$

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where

$$\begin{aligned}
\underset{\sim}{x}_i &= -(M(|Rb| + |R|\delta))_i + m_i(Rb + |Rb|)_i \\
\tilde{x}_i &= (M(|Rb| + |R|\delta))_i + m_i(Rb - |Rb|)_i \\
\alpha_i &= 1 + (|r_i| - r_i)m_i + h_i \\
\beta_i &= 2m_i - 1 - (|r_i| + r_i)m_i - h_i \\
m_i &= M_{ii} \\
r_i &= (I - RA)_{ii} \\
h_i &= (M - MG - I)_{ii}
\end{aligned}$$

and

$$\beta_i \geq \alpha_i \geq 1.$$

Moreover, if $A = I$ *and* $\varrho(\Delta) < 1$, *and if we take* $R := I$ *and* $M := (I - \Delta)^{-1}$, *then the bounds (2) are exact (i.e, achieved).*

The *proof* employs the ideas of the proofs of Theorems 1 and 3 in [2]; details are omitted here.

Comments. The quantities r_i and h_i correct the influence of the approximate inverses R and M; they vanish if $R = A^{-1}$ and $M = (I - G)^{-1} \geq 0$ are used. The last statement of the theorem is Hansen's optimality result [1] as reformulated in [2]. It can be shown that matrices R and $M \geq 0$ satisfying (1) exist if and only if

$$\varrho\left(|A^{-1}|\Delta\right) < 1$$

holds.

References

[1] Hansen E.R., Bounding the solution of interval linear equations, SIAM J. Numer. Anal. 29(1992), 1493–1503

[2] Rohn J., Cheap and tight bounds: the recent result by E. Hansen can be made more efficient, to appear in Interval Computations