Checking Bounds on Solutions of Linear Interval Equations is NP-Hard*

Jiří Rohn

Faculty of Mathematics and Physics
Charles University
Prague, Czech Republic

Dedicated to Prof. M. Fiedler and Prof. V. Pták on the occasion of their retirement

ABSTRACT

We prove that it is NP-hard to decide whether the solution set of a system of linear interval equations is contained in a given interval vector, even in the case that the system matrix is strongly regular.

1. INTRODUCTION

Consider a system of linear interval equations

$$A^I x = b^I \tag{1.1}$$

where

$$A^I = [A_c - \Delta, A_c + \Delta] := \{A'; A_c - \Delta \le A' \le A_c + \Delta\}$$

is an interval matrix and

$$b^I = [b_c - \delta, b_c + \delta] := \{b'; b_c - \delta \le b' \le b_c + \delta\}$$

_____

is an interval vector; here, $A_c, \Delta \in R^{n \times n}$, $b_c, \delta \in R^n$ and $\Delta \geq 0$, $\delta \geq 0$ (matrix and vector inequalities are understood componentwise). The *solution set* of the system (1.1) is defined by

$$X(A^I, b^I) = \{x; A'x = b' \text{ for some } A' \in A^I, b' \in b^I\}.$$

It is well known (cf. e.g. Neumaier [4]) that the solution set is of a complicated nonconvex structure in general. Therefore the problem of solving (1.1) is usually formulated as follows (see [1], [4]): find an interval vector x^I (as narrow as possible) satisfying

$$X(A^I, b^I) \subset x^I \tag{1.2}$$

(provided $X(A^I, b^I)$ is bounded), or verify that $X(A^I, b^I)$ is unbounded. It has been proved recently that this problem is NP-hard [8]. Let us recall that a problem is called NP-hard if each problem in the class NP can be polynomially reduced to it. Thus, unless the class NP of problems solvable by nondeterministic polynomial-time algorithms is equal to the class P of problems solvable by polynomial-time algorithms, which is currently considered highly unlikely (see Garey and Johnson [3] for details), there does not exist a polynomial-time algorithm for solving an NP-hard problem.

In this paper we address another related problem: for a system (1.1), given an n-dimensional interval vector x^I, check whether (1.2) holds, or not. We shall prove that this problem is NP-hard even for a very restricted class of systems with strongly regular interval matrices. Let us recall that an interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ is called *strongly regular* [4] if A_c is nonsingular and

$$\rho(|A_c^{-1}|\Delta) < 1 \tag{1.3}$$

holds (here, ρ denotes the spectral radius and for $A = (a_{ij})$, $|A|$ is defined by $|A| = (|a_{ij}|)$). A well-known result by Beeck [2] states that if A^I satisfies (1.3), then A^I is regular (i.e., each $A \in A^I$ is nonsingular). The problems (1.1) with strongly regular matrices are usually considered "tractable", for several reasons: 1) regularity of A^I can be easily checked (whereas it is an NP-hard problem in general [5]), 2) a vector x^I satisfying (1.2) can be found in polynomial time [4], [7] (whereas the enclosure problem is again NP-hard in general [8]), and 3) several iterative methods (as Rump [10], Rohn [6]) are guaranteed to converge under (1.3) (and convergence is generally not preserved if (1.3) is violated). Nevertheless, the main result of this paper shows that checking bounds on solutions is difficult even in this special case:

THEOREM 1.1. *The following decision problem is NP-hard:*

Instance. *A strongly regular interval matrix A^I and interval vectors*
       b^I, x^I (with rational bounds).
Question. *Is $X(A^I, b^I) \subset x^I$?*

The proof of this theorem will be carried out in section 3, employing two auxiliary results established in section 2. The recent NP-hardness result for computing the optimal (i.e., narrowest) bounds on $X(A^I, b^I)$ (Rohn and Kreinovich [9]) then becomes an easy consequence of Theorem 1.1.

2. AUXILIARY RESULTS

In this section we describe an auxiliary construction and prove its properties to be used later in the proof of the main theorem.

Let A be an arbitrary real nonsingular $n \times n$ matrix and β a positive real number. We shall consider an $(n+1) \times (n+1)$ interval matrix introduced in [9] and defined as follows:

$$A_\beta^I = [A_c - \Delta, A_c + \Delta] \tag{2.1}$$

with

$$A_c = \begin{pmatrix} A^{-1} & 0 \\ (A^{-1})_{n.} & -1 \end{pmatrix} \tag{2.2}$$

and

$$\Delta = \begin{pmatrix} \beta e e^T & 0 \\ 0^T & 0 \end{pmatrix} \tag{2.3}$$

where $(A^{-1})_{n.}$ denotes the n-th row of A^{-1} and $e = (1, 1, \ldots, 1)^T \in R^n$. Let us additionally denote by I the unit matrix and by e_n the vector $(0, 0, \ldots, 0, 1)^T \in R^n$.

PROPOSITION 2.1. *Let A be nonsingular and let β satisfy*

$$0 < \beta < \frac{1}{e^T |A| e}. \tag{2.4}$$

Then the interval matrix A_β^I given by (2.1)-(2.3) is strongly regular and each $A' \in A_\beta^I$ satisfies

$$|(A')^{-1} - A_c^{-1}| \leq \frac{\beta}{1 - \beta e^T |A| e} \begin{pmatrix} |A| e e^T |A| & 0 \\ e^T |A| & 0 \end{pmatrix}. \tag{2.5}$$

Proof. First, a direct computation shows that

$$
\begin{pmatrix} A^{-1} & 0 \\ (A^{-1})_{n.} & -1 \end{pmatrix} \begin{pmatrix} A & 0 \\ e_n^T & -1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0^T & 1 \end{pmatrix},
$$

hence

$$
A_c^{-1} = \begin{pmatrix} A & 0 \\ e_n^T & -1 \end{pmatrix},
$$

so that for $D = |A_c^{-1}|\Delta$ we have

$$
D = \beta \begin{pmatrix} |A|ee^T & 0 \\ e^T & 0 \end{pmatrix}.
$$

Let $Dx' = \lambda x'$ for some (complex) λ and $x' \neq 0$. Let $x' = (x^T, x_{n+1})^T$, then we have

$$
\begin{aligned}
\beta|A|ee^T x &= \lambda x, \qquad\qquad\qquad (2.6) \\
\beta e^T x &= \lambda x_{n+1}.
\end{aligned}
$$

Hence, if $e^T x = 0$, then $\lambda = 0$; if $e^T x \neq 0$, then from (2.6) we obtain $\lambda = \beta e^T |A|e$. Thus $\rho(D) < 1$ due to (2.4), so that A_β^I is strongly regular.

Next, a simple computation gives that $D^2 = \beta(e^T|A|e)D$, hence $D^j = \beta^{j-1}(e^T|A|e)^{j-1}D$ for $j \geq 1$. Then for each $A' \in A_\beta^I$, from the identity $A' = A_c(I - A_c^{-1}(A_c - A'))$, in view of the fact that $\rho(A_c^{-1}(A_c - A')) \leq \rho(D) < 1$ we obtain

$$
\begin{aligned}
|(A')^{-1} - A_c^{-1}| &\leq \sum_1^\infty D^j|A_c^{-1}| = \sum_1^\infty \beta^{j-1}(e^T|A|e)^{j-1}D|A_c^{-1}| \\
&= \frac{\beta}{1 - \beta e^T|A|e} \begin{pmatrix} |A|ee^T|A| & 0 \\ e^T|A| & 0 \end{pmatrix},
\end{aligned}
$$

which proves (2.5). $\square$

In the next proposition we shall consider the solution set of the system

$$
A_\beta^I x = b_\beta^I \qquad\qquad\qquad (2.7)
$$

where A_β^I is as above and b_β^I is the $(n+1)$-dimensional interval vector

$$
b_\beta^I = \left[\begin{pmatrix} -\beta e \\ 0 \end{pmatrix}, \begin{pmatrix} \beta e \\ 0 \end{pmatrix} \right]. \qquad\qquad\qquad (2.8)
$$

Denote $e' = (e^T, 1)^T \in R^{n+1}$.

PROPOSITION 2.2. *Let A be nonsingular, L a nonnegative integer, and let β satisfy*

$$0 < \beta < \frac{1}{\max\{e^T|A|e, L\}}. \tag{2.9}$$

Then

$$z^T A y > L \tag{2.10}$$

holds for some $z, y \in \{-1, 1\}^n$ if and only if the solution set $X(A_\beta^I, b_\beta^I)$ of the system (2.7) does not *satisfy*

$$X(A_\beta^I, b_\beta^I) \subset x^I, \tag{2.11}$$

where $x^I = [\tilde{x}, \tilde{\tilde{x}}]$ is given by

$$\tilde{x} = -\nu e' \tag{2.12}$$

$$\tilde{\tilde{x}} = \left(\nu e^T, \frac{\beta}{1 - \beta L}\right)^T \tag{2.13}$$

and

$$\nu = \frac{\beta \max\{\| A \|_\infty, 1\}}{1 - \beta e^T |A| e}. \tag{2.14}$$

Proof. Let $x \in X(A_\beta^I, b_\beta^I)$, so that $A'x = b'$ for some $A' \in A_\beta^I$, $b' \in b_\beta^I$. Then from Proposition 1 we have

$$
\begin{aligned}
|x| &\leq |A_c^{-1}| \cdot |b'| + |(A')^{-1} - A_c^{-1}| \cdot |b'| \\
&\leq \begin{pmatrix} |A| & 0 \\ e_n^T & 1 \end{pmatrix} \begin{pmatrix} \beta e \\ 0 \end{pmatrix} + \frac{\beta}{1 - \beta e^T |A| e} \begin{pmatrix} |A| e e^T |A| & 0 \\ e^T |A| & 0 \end{pmatrix} \begin{pmatrix} \beta e \\ 0 \end{pmatrix} \\
&\leq \frac{\beta}{1 - \beta e^T |A| e} \begin{pmatrix} |A| e \\ 1 \end{pmatrix} \leq \nu e',
\end{aligned}
$$

hence $X(A_\beta^I, b_\beta^I) \subset [-\nu e', \nu e']$. By comparing this with (2.12) and (2.13), we see that $X(A_\beta^I, b_\beta^I) \subset [\tilde{x}, \tilde{\tilde{x}}]$ does *not* hold if and only if

$$\mu := \max\{x_{n+1}'; x' \in X(A_\beta^I, b_\beta^I)\} > \frac{\beta}{1 - \beta L}.$$

Let $x' \in R^{n+1}$. Then from the construction of A_c and Δ it follows that $x' \in X(A_\beta^I, b_\beta^I)$ if and only if it is of the form $x' = (x^T, x_{n+1})^T$, where

$x \in X([A^{-1} - \beta ee^T, A^{-1} + \beta ee^T], [-\beta e, \beta e])$ and $x_{n+1} = (A^{-1})_{n.}x$. Then from Proposition 2 in [9] we have

$$\mu = \frac{\beta}{1 - \beta \max\{z^T Ay; z, y \in \{-1, 1\}^n\}},$$

hence $\mu > \frac{\beta}{1 - \beta L}$ if and only if $z^T Ay > L$ for some $z, y \in \{-1, 1\}^n$. Hence, (2.10) is true if and only if (2.11) does not hold. □

3. PROOF OF THE MAIN RESULT

We shall now prove the main result formulated in section 1 as a consequence of Proposition 2.

Proof of Theorem 1.1. In [5],Thm. 2.6 it is proved that the decision problem
Instance. A nonsingular rational matrix A and a nonnegative integer L.
Question. Is $z^T Ay > L$ for some $z, y \in \{-1, 1\}^n$?
is NP-hard. We shall polynomially reduce this problem to that one formulated in Theorem 1.1. Given a nonsingular rational matrix A and a nonnegative integer L, choose a rational number β satisfying (2.9), invert A and construct A_β^I, b_β^I and x^I by (2.1)-(2.3), (2.8) and (2.12)-(2.14); this can be done in polynomial time, and A_β^I is strongly regular (Proposition 1). Now, if (2.11) is true, then the solution to the above decision problem is "no", and if (2.11) is false, then the solution is "yes" (Proposition 2). In this way we have polynomially reduced the above-formulated NP-hard problem to that one of Theorem 1.1, hence the latter problem is NP-hard as well. □

In [9] it is proved that computing the narrowest interval vector x_{opt}^I containing the solution set $X(A^I, b^I)$ is NP-hard. This can be now proved as a simple consequence of Theorem 1.1. The narrowest interval vector $x_{opt}^I = [\underline{x}, \overline{x}]$ is obviously given by

$$\begin{aligned} \underline{x}_i &= \min\{x_i; x \in X(A^I, b^I)\}, \\ \overline{x}_i &= \max\{x_i; x \in X(A^I, b^I)\} \end{aligned}$$

$(i = 1, \ldots, n)$. Thus for an arbitrary x^I,

$$X(A^I, b^I) \subset x^I$$

is true if and only if

$$x_{opt}^I \subset x^I$$

holds. Hence, the decision problem of Theorem 1.1 can be polynomially reduced to that of computing x_{opt}^I, which is then NP-hard.

REFERENCES

1 G. Alefeld and J. Herzberger, *Introduction to Interval Computations*, Academic Press, New York, 1983.

2 H. Beeck, Zur Problematik der Hüllenbestimmung von Intervallgleichungssystemen, in: *Interval Mathematics* (K. Nickel, Ed.), Lecture Notes in Comput. Sci. 29, Springer 1975, 150-159.

3 M. E. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, San Francisco, 1979.

4 A. Neumaier, *Interval Methods for Systems of Equations*, Cambridge University Press, Cambridge, 1990.

5 S. Poljak and J. Rohn, Checking robust nonsingularity is NP-hard, *Math. of Control, Signals, and Systems* 6:1-9 (1993).

6 J. Rohn, Systems of linear interval equations, *Lin. Alg. Appls.* 126:39-78 (1989).

7 J. Rohn, Cheap and tight bounds: the recent result by E. Hansen can be made more efficient, to appear in *Interval Computations*.

8 J. Rohn, Enclosing solutions of linear interval equations is NP-hard, to appear in *Computing*.

9 J. Rohn and V. Kreinovich, Computing exact componentwise bounds on solutions of linear systems with interval data is NP-hard, to appear in *SIAM J. Matrix Anal. Appl.*

10 S. M. Rump, Solving algebraic problems with high accuracy, in: *A New Approach to Scientific Computation* (U. Kulisch and W. Miranker, Eds.), Academic Press, New York 1983, 51-120.