NP-hardness results for linear algebraic problems with interval data

Dedicated to my father, Mr. Robert Rohn, in memoriam

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This paper surveys recent results showing that basic linear algebraic problems (such as checking nonsingularity, solving systems of linear equations, matrix inversion etc.) are NP-hard for interval data.

1. INTRODUCTION

It is a well-known fact that basic linear algebraic problems (such as determinant evaluation, checking nonsingularity, solving systems of linear equations, matrix inversion or checking positive definiteness) are polynomial-time since they can be solved by Gaussian elimination which is known to have the property [25]. Although linear algebraic problems with interval data have been studied for 30 years now, only very recently results began to appear showing that most of these problems are NP-hard in their usual interval analytic setting (where "checking" means checking the respective property over all data within given intervals, and "computing" means computing (exact) bounds on the solution over all such data). In this paper we survey the recent developments in this area. In section 2 we sum up results concerning properties of interval matrices, whereas section 3 is devoted to solving linear equations and inverting matrices with interval data. The ideas of the main proofs are explained in section 4. Emphasis is laid on the complexity issues, therefore we do not refer to existing necessary and sufficient conditions or numerical methods as the current literature is considerable and this would lead us beyond the scope of this paper.

The NP-hardness of the problems listed here means that each of them is at least as difficult as the most difficult combinatorial problems belonging to the class NP. Hence, if the well-known conjecture P $\neq$ NP is true (which is currently widely believed to be so), then no NP-hard problem can be solved by a general polynomial-time algorithm (which, of course, does not preclude existence of polynomial algorithms for special classes of instances). We refer to the standard book by Garey and Johnson [8] for the basic concepts of the complexity theory. Throughout the paper, all the data (in particular, the bounds of interval matrices and interval vectors) are assumed to be rational numbers.

2. MATRIX PROPERTIES

Let $\underline{A}, \overline{A}$ be $m \times n$ matrices, $\underline{A} \leq \overline{A}$ (componentwise inequalities). The set of matrices

$$A^I = [\underline{A}, \overline{A}] := \{A;\ \underline{A} \leq A \leq \overline{A}\}$$

is called an interval matrix. In some contexts it is more advantageous to work, instead of $\underline{A}$ and $\overline{A}$, with the center matrix

$$A_c = \frac{1}{2}(\underline{A} + \overline{A})$$

and the nonnegative radius matrix

$$\Delta = \frac{1}{2}(\overline{A} - \underline{A}).$$

Then $A^I = [A_c - \Delta, A_c + \Delta]$. With exception of the last theorem in section 3, we shall consider square interval matrices only, i.e., the case $m = n$.

The NP-hardness results concerning various properties of square interval matrices are grouped here into two theorems. In the first one we consider the general case, whereas in the second one the case of a symmetric interval matrix.

Theorem 2.1 *For a square interval matrix A^I, each of the following problems is NP-hard:*

(i) check whether each $A \in A^I$ is nonsingular,

(ii) compute the "radius of nonsingularity"

$$d(A^I) = \min\{\epsilon \geq 0;\ [A_c - \epsilon\Delta, A_c + \epsilon\Delta] \text{ contains a singular matrix}\},$$

(iii) compute

$$\max\{\det A;\ A \in A^I\},$$

(iv) check whether $\| A \|_2 \leq 1$ for each $A \in A^I$,

(v) check whether each $A \in A^I$ is Hurwitz stable,

(vi) compute the "radius of stability"

$$\min\{\epsilon \geq 0;\ [A_c - \epsilon\Delta, A_c + \epsilon\Delta] \text{ contains an unstable matrix}\},$$

(vii) given a $\lambda \in R^1$, decide whether λ is an eigenvalue of some $A \in A^I$.

The assertion (i) was proved by Poljak and Rohn [14]; we shall describe the main idea of the proof in section 4. (ii), (iii), (vi) and (vii) are consequences of it (the complexity of computing $d(A^I)$ was also studied by Demmel [6]). The assertions (iv) and (v) are due to Nemirovskii [12]. The "min" version of (iii) is obviously also NP-hard. It is worth mentioning that the analogue of the problem (vii) for eigen*vectors* can be solved in polynomial time ([17], Thm. 4.1).

A somewhat related result stating that computing the real structured singular value (introduced by Doyle [7]) is NP-hard was proved by Braatz, Young, Doyle and Morari [3] and independently by Coxson and DeMarco [5].

The second set of results is formulated for symmetric interval matrices. By definition, an interval matrix $A^I = [\underline{A}, \overline{A}]$ is called symmetric if both $\underline{A}$ and $\overline{A}$ are symmetric. Hence, a symmetric interval matrix may contain nonsymmetric matrices. We have these results:

Theorem 2.2 *For a symmetric interval matrix A^I, each of the following problems is NP-hard:*

(i) check whether each symmetric $A \in A^I$ is nonsingular,

(ii) check whether each $A \in A^I$ is positive semidefinite,

(iii) check whether each $A \in A^I$ is positive definite,

(iv) check whether each $A \in A^I$ is Hurwitz stable.

The assertions of this theorem, except (ii), are proved in [21]. (i) and (iv) follow from (iii), whose proof will be sketched in section 4, using characterizations of positive definiteness and stability given in [19]. The assertion (ii) is due to Nemirovskii [12] and its proof employs another technique. We note that the NP-hardness of checking Hurwitz stability of symmetric interval matrices (assertion (iv) here) implies the results for general interval matrices (assertion (v) of Theorem 2.1). We have included both of them as they come from different authors and Nemirovskii's result for general interval matrices has the priority.

3. LINEAR EQUATIONS AND MATRIX INVERSION

Consider a system of linear interval equations

$$A^I x = b^I \tag{1}$$

where A^I is an $n \times n$ interval matrix and b^I is an interval vector (an $n \times 1$ interval matrix). The solution set $X(A^I, b^I)$ defined by

$$X(A^I, b^I) = \{x;\ Ax = b \text{ for some } A \in A^I,\ b \in b^I\} \tag{2}$$

is generally nonconvex (see examples in Neumaier [13]). The exact range of the components of the solution of (1) is described by the numbers

$$\underline{x}_i = \min\{x_i;\ x \in X(A^I, b^I)\}$$

$$\overline{x}_i = \max\{x_i;\ x \in X(A^I, b^I)\} \tag{3}$$

($i = 1, \ldots, n$); they are called the exact (sometimes, optimal) bounds on solution of (1). Since in applications we are interested in *verified* bounds and computation of the exact bounds is usually afflicted with roundoff errors, we are led to the concept of an enclosure: an interval vector x^I is called an enclosure of the solution set if it satisfies

$$X(A^I, b^I) \subset x^I.$$

Such an enclosure, of course, exists only if $X(A^I, b^I)$ is bounded. Various enclosure methods exist to date (see Alefeld and Herzberger [1] or Neumaier [13] for surveys of results), but none of them solves the general problem in polynomial time. This phenomenon is explained by this result:

Theorem 3.1 *The following problem is NP-hard: given square A^I and b^I, compute an enclosure of $X(A^I, b^I)$, or verify that $X(A^I, b^I)$ is unbounded.*

Let us call, as customary, A^I *regular* if each $A \in A^I$ is nonsingular. The proof of Theorem 3.1, given in [20], relies on the fact that if A^I contains at least one nonsingular matrix, then A^I is regular if and only if $X(A^I, b^I)$ is bounded for some (but arbitrary) b^I; hence, an enclosure algorithm can be employed for checking regularity which is NP-hard (Theorem 2.1, (i)).

The previous proof relies heavily on the NP-hardness of checking regularity. But it turns out that computing the exact bounds remains NP-hard even if regularity can be verified. Let us call an interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ *strongly regular* [13] if A_c is nonsingular and

$$\rho(|A_c^{-1}|\Delta) < 1 \tag{4}$$

holds. A well-known result by Beeck [2] states that a strongly regular interval matrix is regular; hence, (4) is a verifiable sufficient regularity condition. In addition to (3), let us also introduce the exact bounds on the inverse of a regular interval matrix by

$$\underline{B}_{ij} = \min\{(A^{-1})_{ij};\ A \in A^I\}$$

$$\overline{B}_{ij} = \max\{(A^{-1})_{ij};\ A \in A^I\} \tag{5}$$

($i, j = 1, \ldots, n$).

Theorem 3.2 *For a strongly regular interval matrix A^I and an interval vector b^I, each of the following problems is NP-hard:*

(i) compute $\underline{x}_i, \overline{x}_i$ ($i = 1, \ldots, n$) given by (3),

(ii) given an interval vector x^I, check whether $X(A^I, b^I) \subset x^I$,

(iii) compute $\underline{B}_{ij}$, $\overline{B}_{ij}$ $(i,j=1,\ldots,n)$ *given by (5),*

(iv) given an interval matrix C^I*, check whether* $A^{-1}\in C^I$ *for each* $A\in A^I$*.*

The result (i) is due to Rohn and Kreinovich [23] and the idea of its proof will be explained in section 4. (ii) is proved in [22] using a similar technique. The assertion (iii) is due to Coxson [4] whose proof is based on the idea of the proof of NP-hardness of checking regularity in [14] and on Theorem 3.1 in [18]. The result (iv) can be derived from the proof of (iii) employing the idea of the proof of (ii).

So far we have assumed that A^I is square. Let us now consider the case of an $m\times n$ interval matrix A^I and an m-dimensional interval vector b^I. The solution set $X(A^I,b^I)$ and the exact bounds $\underline{x}_i,\overline{x}_i$ $(i=1,\ldots,n)$ can be again defined by (2), (3).

Theorem 3.3 *For a general system (1) we have:*

(i) the problem of deciding whether $X(A^I,b^I)\neq\emptyset$ *is NP-complete,*

(ii) computing $\underline{x}_i,\overline{x}_i$ $(i=1,\ldots,n)$ *given by (3) is NP-hard for a system (1) with bounded solution set* $X(A^I,b^I)$*.*

The result (i) was presented by Lakeyev and Noskov in [11] without a proof which was given later in Kreinovich, Lakeyev and Noskov [10]. (ii) was proved also by Kreinovich, Lakeyev and Noskov in [9]. Both proofs use instances with $m>n$ and therefore cannot be applied to the square case.

4. IDEAS OF THE MAIN PROOFS

To explain the techniques used, we shall briefly describe the ideas of the main proofs. A real *symmetric* $n\times n$ matrix $A=(a_{ij})$ is called an MC-matrix if it is of the form

$$a_{ij}\begin{cases} =n & \text{if} \quad i=j\\ \in\{0,-1\} & \text{if} \quad i\neq j\end{cases}$$

$(i,j=1,\ldots,n)$. This definition comes from [21], but the concept of matrices of this form (with a "sufficiently large" M on the diagonal) was first used in [14]. The following result forms the basic link to the complexity theory:

Theorem 4.1 *The following decision problem is NP-complete:*
Instance. *An* $n\times n$ MC*-matrix* A *and a positive integer* L*.*
Question. *Is* $z^TAy\geq L$ *for some* $z,y\in\{-1,1\}^n$*?*

The proof, given by Poljak and Rohn [14], is based on a polynomial reduction of the problem of computing the max-cut in a graph to our problem; max-cut is a known NP-complete problem (Garey and Johnson [8], p. 87).

The result can also be rephrased in this form:

Theorem 4.2 *Computing the number*

$$r(A) = \max\left\{z^T Ay;\ z, y \in \{-1, 1\}^n\right\} \tag{6}$$

is NP-hard for an MC-matrix A.

Let us denote by

$$E = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \\ 1 & \dots & 1 \end{pmatrix}$$

the $n \times n$ matrix of all ones. Then we have the following equivalence:

Theorem 4.3 *Let A be a nonsingular $n \times n$ matrix and L a positive integer. Then*

$$z^T Ay \geq L$$

holds for some $z, y \in \{-1, 1\}^n$ if and only if the interval matrix

$$[A^{-1} - \frac{1}{L}E,\ A^{-1} + \frac{1}{L}E] \tag{7}$$

contains a singular matrix.

This result is proved in [15], Thm. 5.2, assertion (R1). Since an MC-matrix A is diagonally dominant and therefore nonsingular, and can be inverted (by Gaussian elimination) in polynomial time [25], we have a polynomial reduction of the NP-complete problem of Theorem 4.1 to the problem of checking regularity of the interval matrix (7), hence the latter problem is NP-hard; this proves the assertion (i) of Theorem 2.1.

Next, since an MC-matrix A is symmetric and positive definite [21], the same holds for A^{-1}. Then, according to [19], Thm. 3, the symmetric interval matrix (7) contains a singular matrix if and only if it contains a matrix which is not positive definite. Hence we can again use Theorems 4.1 and 4.3 to prove that the problem of checking positive definiteness of each $A \in A^I$ is NP-hard, which is the assertion (iii) of Theorem 2.2. The result for Hurwitz stability (assertion (iv) of Theorem 2.2) follows from Theorem 6 in [19] which states that a symmetric A^I is Hurwitz stable if and only if the symmetric interval matrix $-A^I := \{-A;\ A \in A^I\}$ is positive definite.

Next we turn to linear interval equations. Given a real nonsingular $n \times n$ matrix A, let us consider the system

$$A^I x = b^I \tag{8}$$

where the $(n+1) \times (n+1)$ interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ is given by

$$A_c = \begin{pmatrix} A^{-1} & 0 \\ (A^{-1})_{n.} & -1 \end{pmatrix} \tag{9}$$

and

$$\Delta = \begin{pmatrix} \beta E & 0 \\ 0 & 0 \end{pmatrix} \tag{10}$$

where $(A^{-1})_{n.}$ denotes the nth row of A^{-1} and b^I is given by

$$b^I = \left[\begin{pmatrix} -\beta e \\ 0 \end{pmatrix}, \begin{pmatrix} \beta e \\ 0 \end{pmatrix}\right]$$

where $e = (1, 1, \dots, 1)^T \in R^n$ and β is a positive real parameter. The following result is proved in Rohn and Kreinovich [23]:

Theorem 4.4 *Let A be nonsingular and let β satisfy*

$$0 < \beta < \frac{1}{e^T|A|e}. \tag{11}$$

Then the interval matrix A^I given by (9), (10) is strongly regular and for the system (8) we have

$$\overline{x}_{n+1} = \frac{\beta}{1 - \beta r(A)},$$

where $\overline{x}_{n+1}$ is given by (3) and $r(A)$ by (6).

Hence, this result shows that for an appropriately small β, the value of $r(A)$ can be computed from $\overline{x}_{n+1}$ in polynomial time. Since computing $r(A)$ is NP-hard (Theorem 4.2), the same holds for $\overline{x}_{n+1}$ as well. This proves the assertion (i) of Theorem 3.2. The proofs of (ii)-(iv) are based on similar reasonings [22], [4]. Let us note that the assertion (i) was formulated in [23] only for regular interval matrices; but it is not difficult to verify that the interval matrix A^I defined by (9), (10) and (11) is strongly regular [22].

The results in sections 2 and 3 due to Nemirovskii [12] or Kreinovich, Lakeyev and Noskov [9], [10], [11] are proved by another method using NP-completeness of the problems 3-SATISFIABILITY and PARTITION (Garey and Johnson [8], Theorems 3.1 and 3.5).

5. CONCLUSION

We have shown that many linear algebraic problems with interval data are NP-hard. These results may seem discouraging, but they should not be understood that way. The real message is as follows: if the conjecture P $\neq$ NP is true, then we cannot expect existence of *general* polynomial-time algorithms for solving the problems listed. But this does not preclude existence of very efficient algorithms for solving *special* subclasses of problems frequently occurring in practice (e.g., problems with narrow data intervals). Many such algorithms are known to date: cf. Alefeld and Herzberger [1], Neumaier [13] or Rump [24], among many others. In particular, polynomial-time algorithms usually exist under sign stability assumptions, see e.g. [15], p. 62, [18], Thms. 2.2 and 2.3 and [16], sect. 5. The "bright side" of NP-hardness of interval computations is considered by Traylor and Kreinovich [27].

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