INTERVAL P–MATRICES

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Abstract. A characterization of interval P–matrices is given. The result implies that a symmetric interval matrix is a P–matrix if and only if it is positive definite (although nonsymmetric matrices may be involved). As a consequence it is proved that the problem of checking whether a symmetric interval matrix is a P–matrix is NP–hard.

Key words. interval matrix, P–matrix, positive definiteness, NP–hardness

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1. Introduction. As is well known, an $n \times n$ matrix A is called a P–matrix if all its principal minors are positive. P–matrices play an important role in several areas, e.g. in the linear complementarity theory since they guarantee existence and uniqueness of the solution of a linear complementarity problem (see Murty [6]).

A basic characterization of P–matrices was given by Fiedler and Pták [3]: A is a P–matrix if and only if for each $x \in R^n, x \neq 0$ there exists an i such that $x_i(Ax)_i > 0$ holds. This result immediately implies that a *symmetric* matrix A is a P–matrix if and only if it is positive definite (Wilkinson [13]). In fact, if A is positive definite, then for each $x \neq 0$, from $\sum_i x_i(Ax)_i = x^T Ax > 0$ it follows that $x_i(Ax)_i > 0$ for some i, hence A is a P–matrix; conversely, if A is a P–matrix, then all its leading principal minors are positive, hence it is positive definite in view of the Sylvester determinant criterion [6].

In this paper we focus our attention on interval P–matrices. An interval matrix

$$A^I = [\underline{A}, \overline{A}] = \{A;\ \underline{A} \leq A \leq \overline{A}\},$$

where $\underline{A}$ and $\overline{A}$ are $n \times n$ matrices satisfying $\underline{A} \leq \overline{A}$ (componentwise), is said to be a P–matrix if each $A \in A^I$ is a P–matrix . In section 2 we introduce a finite set of matrices A_z in A^I (whose cardinality is at most 2^{n-1}) such that A^I is a P–matrix if and only if all the matrices A_z are P–matrices (Theorem 2.3)). In view of a similar characterization of positive definiteness of A^I via the matrices A_z (Theorem 2.4), it is then proved in section 3 that a symmetric interval matrix A^I (i.e., with symmetric bounds $\underline{A}$, $\overline{A}$) is a P–matrix if and only if it is positive definite (Theorem 3.2). This is a generalization of the above result for real symmetric matrices, but it is not a simple consequence of it since here nonsymmetric matrices may be involved. As a consequence of this result we obtain that the problem of checking whether a symmetric interval matrix is a P–matrix is NP–hard (Theorem 3.4). This result shows that the exponential number of test matrices A_z used in the necessary and sufficient condition of Theorem 2.3 is highly unlikely to be essentially reducible.

2. Characterizations. Let us introduce an auxiliary set

$$Z = \{z \in R^n;\ z_j \in \{-1, 1\} \text{ for } j = 1, \ldots, n\},$$

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i.e. the set of all ± 1-vectors. The cardinality of Z is obviously 2^n. For an interval matrix

$$A^I = [\underline{A}, \overline{A}],$$

we define matrices $A_z, z \in Z$ by

$$(A_z)_{ij} = \frac{1}{2}(\underline{A}_{ij} + \overline{A}_{ij}) - \frac{1}{2}(\overline{A}_{ij} - \underline{A}_{ij})z_i z_j$$

$(i, j = 1, \ldots, n)$. Clearly, $(A_z)_{ij} = \underline{A}_{ij}$ if $z_i z_j = 1$ and $(A_z)_{ij} = \overline{A}_{ij}$ if $z_i z_j = -1$, hence $A_z \in A^I$ for each $z \in Z$, and the number of mutually different matrices A_z is at most 2^{n-1} (since $A_{-z} = A_z$ for each $z \in Z$), and equal to 2^{n-1} if $\underline{A} < \overline{A}$. The properties in question (P–property and positive definiteness) will be formulated below in terms of the finite set of matrices $A_z, z \in Z$. For a vector $x \in R^n$, let us define its sign vector

$$z = \operatorname{sgn} x$$

by

$$z_i = \begin{cases} 1 & \text{if } x_i \geq 0 \\ -1 & \text{if } x_i < 0 \end{cases}$$

$(i = 1, \ldots, n)$, so that $\operatorname{sgn} x \in Z$. For a matrix $A = (A_{ij})$ we introduce its absolute value by $|A| = (|A_{ij}|)$; a similar notation also applies to vectors.

The basic property of the matrices $A_z, z \in Z$, is summed up in the following auxiliary result; notice that no assumptions on A^I are made.

THEOREM 2.1. *Let A^I be an $n \times n$ interval matrix, $x \in R^n$, and let $z = \operatorname{sgn} x$. Then for each $A \in A^I$ and each $i \in \{1, \ldots, n\}$ we have*

$$x_i(Ax)_i \geq x_i(A_z x)_i. \tag{1}$$

Proof. Let $A \in A^I$ and $i \in \{1, \ldots, n\}$. Then

$$|x_i(Ax)_i - x_i((\frac{1}{2}(\underline{A} + \overline{A}))x)_i| = |x_i((A - \frac{1}{2}(\underline{A} + \overline{A}))x)_i|$$

$$\leq |x_i|(|A - \frac{1}{2}(\underline{A} + \overline{A})| \cdot |x|)_i \leq |x_i|(\frac{1}{2}(\overline{A} - \underline{A})|x|)_i,$$

hence

$$x_i(Ax)_i \geq x_i((\frac{1}{2}(\underline{A} + \overline{A}))x)_i - |x_i|(\frac{1}{2}(\overline{A} - \underline{A})|x|)_i.$$

Since $z = \operatorname{sgn} x$, we have $|x_j| = z_j x_j$ for each j, hence

$$\begin{aligned} x_i(Ax)_i &\geq \sum_j (\frac{1}{2}(\underline{A}_{ij} + \overline{A}_{ij}) - \frac{1}{2}(\overline{A}_{ij} - \underline{A}_{ij})z_i z_j)x_i x_j \\ &= \sum_j (A_z)_{ij} x_i x_j = x_i(A_z x)_i, \end{aligned}$$

which concludes the proof. ∎

As the first consequence of this result, we prove a Fiedler–Pták type characterization of interval P–matrices. Notice that the inequality holds "uniformly" here:

THEOREM 2.2. *An interval matrix A^I is a P–matrix if and only if for each $x \in R^n$, $x \neq 0$, there exists an $i \in \{1, \ldots, n\}$ such that*

$$x_i(Ax)_i > 0 \tag{2}$$

holds for each $A \in A^I$.

Proof. If (2) holds, then each $A \in A^I$ is a P–matrix by the Fiedler-Pták theorem. Conversely, let A^I be a P–matrix and let $x \neq 0$. Put $z = \operatorname{sgn} x$, then A_z is a P–matrix, hence by the Fiedler-Pták theorem we have $x_i(A_z x)_i > 0$ for some i. Then (1) implies $x_i(Ax)_i \geq x_i(A_z x)_i > 0$ for each $A \in A^I$, and we are done. ∎

The following characterization, however, turns out to be much more useful:

THEOREM 2.3. *A^I is a P–matrix if and only if each $A_z, z \in Z$, is a P–matrix.*

Proof. If A^I is a P–matrix, then each $A_z \in A^I$ is obviously also a P–matrix. Conversely, let each $A_z, z \in Z$, be a P–matrix. Let $x \in R^n$, $x \neq 0$, and let $z = \operatorname{sgn} x$. Since A_z is a P–matrix, there exists an i with $x_i(A_z x)_i > 0$, then from Theorem 2.1 we obtain $x_i(Ax)_i \geq x_i(A_z x)_i > 0$ for each $A \in A^I$, hence A^I is a P–matrix by Theorem 2.2. ∎

Another finite characterization of interval P–matrices, formulated in different terms, was proved by Białas and Garloff [1].

In analogy with the terminology introduced for P–matrices, an interval matrix A^I is said to be positive definite if each $A \in A^I$ is positive definite (i.e., satisfies $x^T Ax > 0$ for each $x \neq 0$). The following theorem was proved in [9, Thm. 2]. We give here another proof of this result to make the paper self-contained and to demonstrate that it is a simple consequence of Theorem 2.1:

THEOREM 2.4. *A^I is positive definite if and only if each $A_z, z \in Z$, is positive definite.*

Proof. The "only if" part is obvious since $A_z \in A^I$ for each $z \in Z$. To prove the "if" part, take an $A \in A^I$ and $x \in R^n, x \neq 0$. For $z = \operatorname{sgn} x$, from Theorem 2.1 we have

$$x_i(Ax)_i \geq x_i(A_z x)_i$$

for each i, hence

$$x^T Ax = \sum_i x_i(Ax)_i \geq \sum_i x_i(A_z x)_i = x^T A_z x > 0,$$

so that A is positive definite. Thus, by definition, A^I is positive definite. ∎

The last two theorems reveal that both the P–property and positive definiteness of interval matrices are characterized by the same finite subset of matrices $A_z \in A^I$, $z \in Z$. This relationship will become even more apparent in the case of symmetric interval matrices which we shall consider in the next section.

3. Symmetric interval matrices. For an interval matrix $A^I = [\underline{A}, \overline{A}]$, define an associated interval matrix A^I_s by

$$A^I_s = [\frac{1}{2}(\underline{A} + \underline{A}^T), \frac{1}{2}(\overline{A} + \overline{A}^T)].$$

A^I is called *symmetric* if $A^I = A^I_s$, which is clearly the case if and only if both $\underline{A}$ and $\overline{A}$ are symmetric. Hence, A^I_s is always a symmetric interval matrix. The relationship between positive definiteness and P–property is provided by the following theorem:

THEOREM 3.1. *A^I is positive definite if and only if A^I_s is a P–matrix.*

Proof. For each $z \in Z$, let us denote by A^s_z the matrix A_z for A^I_s, i.e.

$$(A^s_z)_{ij} = \frac{1}{4}(\underline{A}_{ij} + \underline{A}_{ji} + \overline{A}_{ij} + \overline{A}_{ji}) - \frac{1}{4}(\overline{A}_{ij} + \overline{A}_{ji} - \underline{A}_{ij} - \underline{A}_{ji})z_i z_j$$

$(i, j = 1, \dots, n)$. Then A^s_z is symmetric and a direct computation shows that

$$x^T A^s_z x = x^T A_z x \tag{3}$$

holds for each $x \in R^n$. Now, if A^I is positive definite, then each $A_z, z \in Z$ is positive definite, hence each A^s_z is positive definite due to (3), so that A^s_z is a P–matrix, hence A^I_s is a P–matrix by Theorem 2.3. Conversely, if A^I_s is a P–matrix, then each $A^s_z, z \in Z$ is a P–matrix, hence it is positive definite due to its symmetry, thus each $A_z, z \in Z$ is positive definite by (3) and A^I is positive definite by Theorem 2.4. $\square$

Our main result on symmetric interval matrices is now obtained as a simple consequence of Theorem 3.1.

THEOREM 3.2. *A symmetric interval matrix A^I is a P–matrix if and only if it is positive definite.*

Proof. The result follows immediately from Theorem 3.1 since a symmetric interval matrix A^I satisfies $A^I = A^I_s$ by definition. $\square$

At the beginning of the Introduction we showed that a real symmetric matrix is a P–matrix if and only if it is positive definite. The result of Theorem 3.2 sounds verbally alike, but it is not a simple consequence of the real case since here nonsymmetric matrices may be involved. In fact, it can be immediately seen that a symmetric interval matrix $A^I = [\underline{A}, \overline{A}]$ contains nonsymmetric matrices if and only if $\underline{A}_{ij} < \overline{A}_{ij}$ holds for some $i \neq j$.

An interval matrix A^I is called regular (cf. Neumaier [7]) if each $A \in A^I$ is nonsingular. The following result shows that for symmetric interval matrices the P–property is preserved by regularity. Several other results of this type are summed up in [10].

THEOREM 3.3. *A symmetric interval matrix A^I is a P–matrix if and only if it is regular and contains at least one symmetric P–matrix.*

Proof. A symmetric interval P–matrix A^I is regular (each $A \in A^I$ has a positive determinant) and contains a symmetric P–matrix $\underline{A}$. If A^I is regular and contains a symmetric P–matrix A_0, then A_0 is positive definite, hence A^I is positive definite by Theorem 3 in [9], which in the light of Theorem 3.2 means that A^I is a P–matrix. $\square$

Another relationship between regularity and P–property of interval matrices was established in [8, Thm. 5.1, assert. (B1)]: an interval matrix $A^I = [\underline{A}, \overline{A}]$ is regular if and only if $(\underline{A} + \overline{A} - S(\overline{A} - \underline{A}))^{-1}(\underline{A} + \overline{A} + S(\overline{A} - \underline{A}))$ is a P–matrix for each signature matrix S (i.e., a diagonal matrix with ± 1 diagonal elements). This topic was recently studied by Johnson and Tsatsomeros [5].

The necessary and sufficient condition of Theorem 2.3 employs up to 2^{n-1} test matrices $A_z, z \in Z$. There is a natural question whether an essentially simpler criterion can be found. The following Theorem 3.4 gives an indirect answer to this question: it implies that an existence of a polynomial-time algorithm for checking the P–property of symmetric interval matrices would imply that the complexity classes P and NP are equal, thereby running contrary to the current (unproved) conjecture that P$\neq$NP. We refer the reader to the classical book by Garey and Johnson [4] for a detailed discussion of the problem "P=NP" and related issues.

THEOREM 3.4. *The following problem is NP-hard:*
Instance. *A symmetric interval matrix* $A^I = [\underline{A}, \overline{A}]$ *with rational bounds* $\underline{A}, \overline{A}$.
Question. *Is* A^I *a* P*–matrix?*

Proof. By Theorem 3.2, A^I is a P–matrix if and only if it is positive definite; checking positive definiteness of symmetric interval matrices was proved to be NP-hard in [11]. $\square$

Coxson [2] proved that the P–matrix problem for real matrices is co-NP-complete. His result concerns nonsymmetric matrices since the symmetric case can be solved by Sylvester determinant criterion which can be performed in polynomial time (Schrijver [12]). Theorem 3.4 shows that for interval matrices even the symmetric case is NP-hard.

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