NP-Hardness Results for Some Linear and Quadratic Problems*

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Abstract

Several problems concerning norms, linear inequalities, linear equations, linear programming and quadratic programming are proved to be NP-hard.

Key words. Norm, linear inequalities, linear equations, linear programming, quadratic programming, NP-hardness

1 Introduction

The first part of this report (sections 2 to 5) was originally made as a transcript of transparencies of seminar talks[1]. Improvements and consequences found shortly after the transcription had been completed were added as Appendices 1 to 4. In this rather incoherent form, the main result is Theorem 2, supported by Proposition 2 (already known in a slightly different setting). Among other consequences, it is shown that computing $\|A\|_{\infty,1}$ within accuracy $\frac{1}{2}$ is NP-hard (Corollary 9), which in turn implies that the same is true for computing the maximal value of a convex quadratic program (Corollary 11) and for one of the two bounds on the optimal value of a linear program with inexact right-hand side (Corollary 12). Another result (Corollary 3) shows that checking sensitivity of a system of linear equations is an NP-hard problem.

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2 MC-matrices

The following concept will be used as a basic tool throughout this report:

Definition A real *symmetric* $n \times n$ matrix $A = (a_{ij})$ is called an MC-matrix[2] if it is of the form

$$a_{ij} \begin{cases} = n & \text{if } \; i = j \\ \in \{0, -1\} & \text{if } \; i \neq j \end{cases}$$

$(i, j = 1, \ldots, n)$.

Proposition 1 *If A is an MC–matrix, then A^{-1} is nonnegative and symmetric positive definite.*

Proof. By definition, A is of the form

$$A = nI - A_0 = n(I - \frac{1}{n}A_0)$$

where $A_0 \geq 0$ and $\|\frac{1}{n}A_0\|_\infty \leq \frac{n-1}{n} < 1$, hence

$$A^{-1} = \frac{1}{n}\sum_{0}^{\infty}(\frac{1}{n}A_0)^j \geq 0.$$

A is symmetric by definition; it is positive definite since for $x \neq 0$,

$$x^T Ax \geq n\|x\|_2^2 - \sum_{i \neq j} |x_i x_j| = (n+1)\|x\|_2^2 - \|x\|_1^2 \geq \|x\|_2^2 > 0.$$

Hence A^{-1} is also symmetric and positive definite. □

The next result is due to Poljak and Rohn [8] (given there in a slightly different formulation without using the concept of an MC-matrix). We add the proof for completeness.

Proposition 2 *The following decision problem is NP-complete:*
Instance. *An MC-matrix A and a positive integer L.*
Question. *Is $z^T Az \geq L$ for some $z \in \{-1, 1\}^n$?*

Proof. Let (N, E) be a graph with $N = \{1, \ldots, n\}$. Let $A = (a_{ij})$ be given by

$$a_{ij} = \begin{cases} n & \text{if } i = j \\ 0 & \text{if } \{i, j\} \notin E, \; i \neq j \\ -1 & \text{if } \{i, j\} \in E, \; i \neq j \end{cases}$$

[2] from ”maximum cut”; explained in the proof of Proposition 2

then A is an MC–matrix. For $S \subseteq N$, define a cut by

$$c(S) = \text{Card}\{\{i,j\} \in E;\ \text{exactly one of } i,j \text{ is in } S\}.$$

If z is given by

$$z_k = \begin{cases} 1 & \text{if } k \in S \\ -1 & \text{if } k \notin S \end{cases}$$

then

$$c(S) = \frac{1}{4}(z^T Az + 2\text{Card}(E) - n^2),$$

hence

$$c(S) \geq L$$

if and only if

$$z^T Az \geq 4L - 2\text{Card}(E) + n^2.$$

Since the problem

$$\text{"}c(S) \geq L\text{"}$$

(maximum cut in a graph) is NP-complete (Garey and Johnson [1]), the current problem is NP-hard. It is obviously in the class NP, since a guessed solution z can be verified in polynomial time; hence it is NP-complete. $\square$

3 The result

Theorem 1 below forms a common basis for several NP-hardness results listed in the next section.

Proposition 3 *Let A be an MC–matrix and L a positive integer. Then*

$$z^T Az \geq L$$

holds for some $z \in \{-1,1\}^n$ if and only if the system

$$-e \leq LA^{-1}x \leq e$$

has a solution satisfying

$$\|x\|_1 \geq 1$$

(where $e = (1,1,\ldots,1)^T$ and $\|x\|_1 = \sum_i |x|_i$).

Proof. $\Rightarrow$: Let $z^T Az \geq L$. Put

$$x = \frac{Az}{z^T Az}$$

then

$$|LA^{-1}x| = \left|\frac{Lz}{z^TAz}\right| \leq |z| = e$$

and

$$\|x\|_1 = \frac{e^T|Az|}{z^TAz} = \frac{z^TAz}{z^TAz} = 1.$$

$\Leftarrow$: If $|LA^{-1}x| \leq e$ and $\|x\|_1 \geq 1$, then for z given by $z_i = 1$ if $x_i \geq 0$ and $z_i = -1$ otherwise we have

$$L \leq L\|x\|_1 = Lz^Tx = Lz^TAA^{-1}x \leq |z^TA|e = z^TAz.$$

□

Theorem 1 *The following decision problem is NP–complete:*
Instance. *A nonnegative symmetric positive definite rational matrix A.*
Question. *Does the system*

$$-e \leq Ax \leq e$$

(where $e = (1, 1, \ldots, 1)^T$) have a solution satisfying

$$\|x\|_1 \geq 1 \text{ ?}$$

Proof. According to Propositions 2 and 3, the NP-complete problem

$$\text{"}z^TAz \geq L\text{"}$$

can be polynomially reduced to this one (if A is an MC–matrix, then LA^{-1} is nonnegative symmetric positive definite), hence the current problem is NP-hard.

If the problem has a solution, then it also has a rational solution of the form

$$x = \frac{Az}{z^TAz}$$

(proof of Proposition 3) which can be checked in polynomial time; thus the problem belongs to the class NP, hence it is NP-complete. □

4 Corollaries

The following five corollaries are direct consequences of Theorem 1. The instances are always assumed to be rational without further notice.

Corollary 1 *The following problem is NP-hard:*
Instance. $A \in R^{m\times n},\ b \in R^m,\ m \geq 2n,$ *L positive integer.*
Question. *Does each solution of the system*

$$Ax \leq b$$

satisfy

$$\|x\|_1 < L \text{ ?}$$

Corollary 2 *The following problem is NP-hard:*
Instance. $A, B \in R^{n \times n},\ b \in R^n$.
Question. *Does the system*

$$Ax + B|x| \le b$$

have a solution?

Corollary 3 *The following problem is NP-hard:*
Instance. *A nonnegative symmetric positive definite* $A \in R^{n\times n},\ b \in R^n,\ \delta > 0,$ $\epsilon > 0$*; denote* $x = A^{-1}b$*.*
Question. *Does the solution of each* $Ax' = b'$ *with* $\|b' - b\|_\infty < \delta$ *satisfy* $\|x' - x\|_1$ $< \epsilon$ *?*

Corollary 4 *For* $A \in R^{m\times n},\ b \in R^m,\ c \in R^n,\ m \ge 2n$*, it is NP-hard to compute*

$$\max\{c^T|x|;\ Ax \le b\}.$$

Note A linear programming problem with objective $c^T x$ can be solved in polynomial time (Khachiyan [6]).

Corollary 5 *For a symmetric positive definite* $A \in R^{n\times n}$ *and* $a, b \in R^n$*, it is NP-hard to compute the optimal value of the quadratic programming problem*

$$\max\{x^T Ax;\ a \le x \le b\}.$$

Note NP-hardness of quadratic programming with indefinite matrices was proved by Murty and Kabadi [7].
The *proofs* follow directly from Theorem 1 and Proposition 2.

5 Nearness to singularity

Let us use the norm (Golub and van Loan [3])

$$\|A\|_{1,\infty} = \max_{i,j} |a_{ij}|.$$

The number

$$d(A) = \min\{\|A - A'\|_{1,\infty};\ A' \text{ singular}\}$$

is called the componentwise distance to the nearest singular matrix (Demmel [4]). If A is rational, then $d(A)$ is rational [8].

Corollary 6 *Suppose there exists a polynomial-time algorithm which for each* $n \times n$ *nonnegative symmetric positive definite rational matrix* A *computes a rational approximation* $d'(A)$ *of* $d(A)$ *satisfying*

$$|d'(A) - d(A)| < \frac{1}{12n^4}$$

Then P=NP.

Proof. A direct computation shows that for an MC-matrix A we have

$$\frac{1}{12n^4} \leq \frac{d^2(A^{-1})}{d(A^{-1})+2}$$

hence

$$|d'(A^{-1}) - d(A^{-1})| < \frac{d^2(A^{-1})}{d(A^{-1})+2}$$

which implies that

$$z^T A z \geq L$$

holds for some $z \in \{-1,1\}^n$ if and only if

$$\left\lceil \frac{1}{d'(A^{-1})} + \frac{1}{2} \right\rceil \geq L.$$

Hence, if such a polynomial-time algorithm exists, then P=NP. □

6 Appendix 1: $\|A\|_{\infty,1}$

The material of this appendix was found later, when the previous part had been already written. In my view, Theorem 2 below forms the core of this report, as it clarifies the relationship between Proposition 2, Theorem 1, Corollary 5 and Corollary 6, and offers a deeper insight into the matter[3]. We shall use the norm

$$\|A\|_{\infty,1} = \max\{\|Ax\|_1;\ \|x\|_\infty = 1\}$$

(see [3, p. 15]; $\|x\|_\infty = \max_i |x_i|$).

Theorem 2 *For an MC-matrix A we have*

$$\begin{aligned}
\|A\|_{\infty,1} &= \max\{z^T A z;\ z \in \{-1,1\}^n\} \\
&= \max\{x^T A x;\ -e \leq x \leq e\} \\
&= \max\{\|x\|_1;\ -e \leq A^{-1}x \leq e\} \\
&= \frac{1}{\min\{x^T A^{-1} x;\ \|x\|_1 = 1\}} \\
&= \frac{1}{d(A^{-1})}.
\end{aligned}$$

Proof. 1) If $\|x\|_\infty = 1$, then x belongs to the unit cube $[-1,1]^n$ and therefore can be expressed as a convex combination of its vertices which are just the points in $\{-1,1\}^n$. Hence from convexity of the norm we have

$$\|A\|_{\infty,1} = \max\{\|Az\|_1;\ z \in \{-1,1\}^n\} = \max\{z^T A z;\ z \in \{-1,1\}^n\}$$

[3]another applications of Theorem 2 are given in appendices 2 and 3

(since $\|Az\|_1 = e^T|Az| = z^TAz$ for an MC-matrix A and $z \in \{-1,1\}^n$).

2) x^TAx is convex (since A is positive definite), hence its maximum value over the cube $\{x;\ -e \le x \le e\}$ is achieved at some of its vertices, implying

$$\max\{x^TAx;\ -e \le x \le e\} = \max\{z^TAz;\ z \in \{-1,1\}^n\} = \|A\|_{\infty,1}.$$

3) Since an MC-matrix A is nonsingular, we have

$$\max\{\|x\|_1;\ -e \le A^{-1}x \le e\} = \max\{\|Ay\|_1;\ -e \le y \le e\} =$$
$$\max\{\|Ay\|_1;\ \|y\|_\infty \le 1\} = \max\{\|Ay\|_1;\ \|y\|_\infty = 1\} = \|A\|_{\infty,1}.$$

4) For a positive real number λ,

$$\|A\|_{\infty,1} \ge \lambda$$

holds iff $|A^{-1} - A'| \le \frac{1}{\lambda}ee^T$ for some A' which is not positive definite [11, proof, equivalence 0) $\Leftrightarrow$ 1)] iff $x'^TA^{-1}x' - \frac{1}{\lambda}|x'|^Tee^T|x'| = x'^TA^{-1}x' - \frac{1}{\lambda}\|x'\|_1^2 \le 0$ for some $x' \ne 0$ iff $x^TA^{-1}x \le \frac{1}{\lambda}$ for some x with $\|x\|_1 = 1$ iff

$$\frac{1}{\min\{x^TA^{-1}x;\ \|x\|_1 = 1\}} \ge \lambda,$$

which gives

$$\|A\|_{\infty,1} = \frac{1}{\min\{x^TA^{-1}x;\ \|x\|_1 = 1\}}.$$

5) By Kahan's theorem [5, p. 775],

$$\|A\|_{\infty,1} = \frac{1}{\min\{\|A^{-1} - A'\|_{1,\infty};\ A' \text{ singular}\}} = \frac{1}{d(A^{-1})}.$$

□

Corollary 7 *Computing $\|A\|_{\infty,1}$ is NP-hard for MC-matrices.*

Proof. From Proposition 2 and Theorem 2. □

Corollary 8 *The following problem is NP-hard:*
Instance. *A symmetric rational M-matrix A.*
Question. *Is $\|A\|_{\infty,1} \ge 1$?*

Proof. For an MC-matrix A, $z^TAz \ge L$ holds if and only if $\|\frac{1}{L}A\|_{\infty,1} \ge 1$, where $\frac{1}{L}A$ is an M-matrix. Hence the problem of Proposition 2 can be polynomially reduced to this one. □

The NP-hardness part of Theorem 1 follows from this result and from Theorem 2.

Corollary 9 *Suppose there exists a polynomial-time algorithm which for each MC-matrix A computes a rational number $\nu(A)$ satisfying*

$$|\nu(A) - \|A\|_{\infty,1}| < \frac{1}{2}.$$

Then P=NP.

Proof. If such an algorithm exists, then $\|A\|_{\infty,1} = [\nu(A) + \frac{1}{2}]$ (since $\|A\|_{\infty,1}$ is integer for an MC-matrix A), hence the NP-hard problem of Corollary 7 can be solved in polynomial time, implying P=NP. □

In the next corollary we present a problem whose complexity depends on the norm used:

Corollary 10 *The decision problem*

Instance. *A nonnegative symmetric positive definite rational matrix A.*

Question. *Is $x^T Ax \leq 1$ for some x with $\|x\| = 1$?*

is NP-complete if the norm $\|\cdot\|_1$ is used and is solvable in polynomial time for $\|\cdot\|_2$.

Proof. NP-hardness of the problem for $\|\cdot\|_1$ follows from Proposition 2 and Theorem 2. The fact that it belongs to NP is proved via a similar construction as in Proposition 3 (see [11]). $x^T Ax \leq 1$ for some x with $\|x\|_2 = 1$ holds if and only if $x^T(A-I)x \leq 0$ for some $x \neq 0$, which is the case if and only if $A - I$ is not positive definite. Since $A - I$ is symmetric, the latter fact can be verified in polynomial time using Sylvester determinant criterion and Gaussian elimination. □

The last result shows that the norm $\|A\|_{\infty,1}$ has nontrivial properties and is worth further studying. It is preceded by a "theorem on the alternative" which may be of independent interest:

Proposition 4 *Let $A, B \in R^{n\times n}$, A nonsingular, $B \geq 0$. Then exactly one of the two alternatives holds:*

(i) the inequality $B|Ax| \geq |x|$ has a nonzero solution,

(ii) the inequality $B|Ax| < |x|$ has a solution in each orthant.

Proof. 1) $B|Ax| \geq |x|$ for some $x \neq 0$ iff $B|x'| \geq |A^{-1}x'|$ for some $x' \neq 0$ iff

$$|A' - A^{-1}| \leq B$$

for some singular A' [10, Lemma 2.1].

2) $B|Ax| < |x|$ has a solution in each orthant iff each A' satisfying

$$|A' - A^{-1}| \leq B$$

is nonsingular [9, Thm. 3].

Clearly, exactly one of the two possibilities occurs. □

Proposition 5 *A nonsingular matrix A satisfies $\|A\|_{\infty,1} < 1$ if and only if in each orthant there exists an x satisfying $\|Ax\|_1 < 1$ and $|x| \ge e$.*

Proof. For $B = ee^T$, $B|Ax| \ge |x|$ is equivalent to $\|Ax\|_1 \ge \|x\|_\infty$, hence $B|Ax| \ge |x|$ has a nonzero solution iff $\|A\|_{\infty,1} \ge 1$. Thus $\|A\|_{\infty,1} < 1$ holds iff

$$\|Ax'\|_1 e < |x'|$$

has a solution in each orthant. Setting $x = \frac{x'}{\min_i |x'_i|}$, we see that this is equivalent to the fact that

$$\|Ax\|_1 < 1$$
$$|x| \ge e$$

has a solution in each orthant. □

7 Appendix 2: Approximate quadratic programming is NP-hard

The results of the previous section enable us to strengthen the formulation of Corollary 5:

Corollary 11 *Suppose there exists a polynomial-time algorithm which for each integer data A, b, c, A symmetric positive definite, computes a rational number $\nu(A, b, c)$ satisfying*

$$|\nu(A, b, c) - \max\{x^T Ax + c^T x;\ 0 \le x \le b\}| < \frac{1}{2}.$$

Then P=NP.

Proof. Due to Theorem 2, for an MC-matrix A we have

$$\|A\|_{\infty,1} = \max\{x^T Ax;\ -e \le x \le e\} = \max\{y^T Ay - 2(Ae)^T y;\ 0 \le y \le 2e\} + e^T Ae,$$

hence

$$|\nu(A, 2e, -2Ae) + e^T Ae - \|A\|_{\infty,1}| < \frac{1}{2}$$

and the conclusion follows from Corollary 9. □

8 Appendix 3: Linear programming with inexact right-hand side is NP-hard

For a linear programming problem

$$\text{minimize } c^T x$$

subject to

$$Ax = b,$$

$$x \geq 0,$$

denote

$$f(A, b, c) = \inf\{c^T x;\ Ax = b, x \geq 0\}$$

(so that $f = -\infty$ if the problem is unbounded and $f = \infty$ if it is infeasible). Consider the problem with the right-hand side ranging within the bounds $\underline{b}$ and $\overline{b}$ (componentwise). With A and c fixed, define

$$\underline{f} = \inf\{f(A, b, c);\ \underline{b} \leq b \leq \overline{b}\}$$

$$\overline{f} = \sup\{f(A, b, c);\ \underline{b} \leq b \leq \overline{b}\}.$$

Obviously,

$$\underline{f} = \inf\{c^T x;\ \underline{b} \leq Ax \leq \overline{b}, x \geq 0\},$$

hence $\underline{f}$ can be determined by solving an LP problem, which can be done in polynomial time [6]. But the case of $\overline{f}$ is different:

Corollary 12 *Computing $\overline{f}$ within accuracy $\frac{1}{2}$ is NP-hard for rational data $A, \underline{b}, \overline{b}, c$ and for a finite value of $\overline{f}$.*

Proof. For an MC-matrix A, consider the problem

$$\min\{e^T x_1 + e^T x_2;\ (A^{-1})^T x_1 - (A^{-1})^T x_2 = b, x_1 \geq 0, x_2 \geq 0\}$$

with

$$-e \leq b \leq e.$$

From the duality theorem and Theorem 2 we have

$$\overline{f} = \sup_{-e \leq b \leq e} \max\{b^T y;\ -e \leq A^{-1} y \leq e\} = \max\{e^T |y|;\ -e \leq A^{-1} y \leq e\} = \|A\|_{\infty,1}$$

and it suffices to apply Corollary 9. □

Note A linear programming problem with the right-hand side satisfying $\underline{b} \leq b \leq \overline{b}$ can be also viewed as a parametric linear programming problem with fully parametrized right-hand side. Hence this problem is also NP-hard.

9 Appendix 4: Complexity of solving linear interval inequalities

Under a system of linear interval inequalities $A^I x \leq b^I$ we understand the family of systems of linear inequalities

$$Ax \leq b,$$
$$A \in A^I, \; b \in b^I,$$

where $A^I = \{A;\ \underline{A} \leq A \leq \overline{A}\}$ is an $m \times n$ interval matrix and $b^I = \{b;\ \underline{b} \leq b \leq \overline{b}\}$ is an interval m-vector. There are two basic problems concerning solvability of such families of systems: first, whether *each* system $Ax \leq b$ with data satisfying $A \in A^I, b \in b^I$ has a solution; second, whether *some* of such systems has a solution.

The first problem was solved by Rohn and Kreslová [12]: *each* system $Ax \leq b, A \in A^I, b \in b^I$ has a solution if and only if the system of linear inequalities

$$\overline{A}x_1 - \underline{A}x_2 \leq \underline{b}$$

$$x_1 \geq 0, \; x_2 \geq 0$$

has a solution. Since this can be checked by solving an associated linear programming problem, the first problem can be solved in polynomial time [6].

Rather surprisingly, it turns out that the second problem is more involved. For a square matrix A,

$$-e \leq Ax \leq e,$$
$$\|x\|_1 \geq 1$$

is equivalent to

$$\begin{pmatrix} A \\ -A \\ 0^T \end{pmatrix} x - \begin{pmatrix} 0 \\ 0 \\ e^T \end{pmatrix} |x| \leq \begin{pmatrix} e \\ e \\ -1 \end{pmatrix}$$

which, due to the theorem by Gerlach [2], is the case if and only if x solves

$$A'x \leq b'$$

for *some* $A' \in A^I, b' \in b^I$, where

$$A^I = \left[\begin{pmatrix} A \\ -A \\ -e^T \end{pmatrix}, \begin{pmatrix} A \\ -A \\ e^T \end{pmatrix} \right],$$

$$b^I = \left[\begin{pmatrix} e \\ e \\ -1 \end{pmatrix}, \begin{pmatrix} e \\ e \\ -1 \end{pmatrix} \right].$$

Hence, the *second* problem is NP-hard in view of Theorem 1. It is even NP-complete, since for guessed A and b, solvability of $Ax \leq b$ can be checked in polynomial time [6].

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