Linear Interval Equations: Computing Sufficiently Accurate Enclosures is NP-Hard*

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Abstract

It is proved that if there exists a polynomial-time algorithm which for each system of linear interval equations with a strongly regular $n \times n$ interval matrix computes an enclosure of the solution set with absolute accuracy better than $\frac{1}{4n^4}$, then P=NP.

Key words. Linear interval equations, enclosure, NP-hardness

1 Introduction

This report is partly a transcript of a poster[1]. The main result (Theorem 1) shows that one of the basic problems in validated computations is more difficult than expected.

2 Enclosures

For a system of linear interval equations

$$A^I x = b^I \tag{1}$$

(A^I square), *enclosure* is defined as an interval vector $[\underline{y}, \overline{y}]$ satisfying

$$X \subseteq [\underline{y}, \overline{y}]$$

*This work was supported in part by the Czech Republic Grant Agency under grant GAČR 201/93/0429

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[1]presented at the international workshop Applications of Interval Computations, El Paso, Texas, February 1995 (sections 2 to 4)

where X is the solution set:

$$X = \{x;\ Ax = b \text{ for some } A \in A^I,\ b \in b^I\}.$$

If A^I is regular, then there exists the narrowest (or: optimal) enclosure $[\underline{x}, \overline{x}]$ given by

$$\underline{x}_i = \min_X x_i,$$
$$\overline{x}_i = \max_X x_i$$

for each i. Computing $[\underline{x}, \overline{x}]$ was proved to be NP-hard (Rohn and Kreinovich [5]). But it turns out that the same is true for computing "sufficiently accurate" enclosures:

3 The result

Theorem 1 *Suppose there exists a polynomial-time algorithm which for each strongly regular $n \times n$ interval matrix A^I and each b^I (both with rational bounds) computes a rational enclosure $[\underline{y}, \overline{y}]$ of X satisfying*

$$\overline{x}_i \le \overline{y}_i \le \overline{x}_i + \frac{1}{4n^4} \tag{2}$$

for each i. Then P=NP.

4 Comments

$A^I = [A_c - \Delta, A_c + \Delta]$ is called strongly regular if $\varrho(|A_c^{-1}|\Delta) < 1$ (a well-known sufficient regularity condition).

P and NP are the well-known complexity classes. The conjecture that P$\neq$NP, although unproved, is widely believed to be true (Garey and Johnson [1]).

Hence, the problem of computing sufficiently accurate enclosures is by far more *difficult* than previously believed: an existence of a polynomial-time algorithm yielding the accuracy (2) would imply polynomial-time solvability of all problems in the class NP, thereby making an enormous breakthrough in theoretical computer science.

5 Proof

1) Denote $e = (1, 1, \ldots, 1)^T \in R^n$ and $Z = \{z \in R^n;\ |z| = e\}$, so that Z is the set of all ± 1-vectors. We shall use matrix norms

$$\|A\|_s = e^T|A|e = \sum_i \sum_j |a_{ij}|$$

and

$$\|A\|_{\infty,1} = \max\{\|Az\|_1;\ z \in Z\} \tag{3}$$

(where $\|x\|_1 = \sum_i |x_i|$; cf. [2]). $[\alpha]$ denotes the integer part of a real number α.

2) A real symmetric $n \times n$ matrix $A = (a_{ij})$ is called an MC-matrix if it is of the form

$$a_{ij} \begin{cases} = n & \text{if } \ i = j \\ \in \{0, -1\} & \text{if } \ i \neq j \end{cases}$$

$(i, j = 1, \ldots, n)$. For an MC-matrix A we obviously have

$$n \leq e^T A e \leq \|A\|_{\infty,1} \leq \|A\|_s \leq n(2n-1). \tag{4}$$

Also,

$$z_i (Az)_i > 0 \tag{5}$$

holds for each $z \in Z$ and each $i \in \{1, \ldots, n\}$. We shall essentially use the fact that computing $\|A\|_{\infty,1}$ is NP-hard for MC-matrices [3, Thm. 2.6]. In the sequel we shall construct, for a given $n \times n$ MC-matrix A, a linear interval system with interval matrix of size $3n \times 3n$ such that if $\overline{y}_i$ satisfies (2), then

$$\|A\|_{\infty,1} = [\|A\|_s + 2 - \frac{1}{\overline{y}_i}].$$

Hence, if such a $\overline{y}_i$ can be computed in polynomial time, then $\|A\|_{\infty,1}$ can also be computed in polynomial time and since this is an NP-hard problem, P=NP will follow.

3) For a given $n \times n$ MC-matrix A (which is diagonally dominant and therefore nonsingular), consider a linear interval system

$$A^I x = b^I \tag{6}$$

with $A^I = [A_c - \Delta, A_c + \Delta]$, $b^I = [b_c - \delta, b_c + \delta]$ given by

$$A_c = \begin{pmatrix} 0 & -I & 0 \\ -I & 0 & A^{-1} \\ 0 & A^{-1} & A^{-1} \end{pmatrix},$$

$$\Delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta e e^T \end{pmatrix}$$

(all the blocks are $n \times n$, I is the unit matrix),

$$b_c = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\delta = \begin{pmatrix} 0 \\ 0 \\ \beta e \end{pmatrix}$$

(all the blocks are $n \times 1$) and

$$\beta = \frac{1}{\|A\|_s + 2}. \tag{7}$$

We shall first prove that A^I is strongly regular. Since

$$A_c^{-1} = \begin{pmatrix} A^{-1} & -I & I \\ -I & 0 & 0 \\ I & 0 & A \end{pmatrix}$$

(as it can be easily verified), we have

$$|A_c^{-1}|\Delta = \begin{pmatrix} 0 & 0 & \beta ee^T \\ 0 & 0 & 0 \\ 0 & 0 & \beta|A|ee^T \end{pmatrix}.$$

This matrix has eigenvalues $\lambda = 0$ (multiple) and $\lambda = \beta\|A\|_s$. Hence $\varrho(|A_c^{-1}|\Delta) = \beta\|A\|_s < 1$ due to (7), and A^I is strongly regular.

4) For the linear interval system (6), consider a solution x satisfying $\tilde{A}x = \tilde{b}$ for some $\tilde{A} \in A^I$, $\tilde{b} \in b^I$. If we decompose x as

$$x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix},$$

then we have

$$\begin{aligned} x^2 &= 0 \\ x^1 &= A^{-1}x^3 \\ A'x^3 &= b' \end{aligned}$$

for some A', b' satisfying $|A^{-1} - A'| \le \beta ee^T$ and $|b'| \le \beta e$, hence x^3 is a solution of the linear interval system

$$[A^{-1} - \beta ee^T, A^{-1} + \beta ee^T]x' = [-\beta e, \beta e] \tag{8}$$

whose matrix is obviously again strongly regular. From [4, Thm. 2.2] we have that for each $z \in Z$ the equation

$$A^{-1}x = \beta(\|x\|_1 + 1)z \tag{9}$$

has a unique solution x_z. A direct substitution shows that the solution has the form

$$x_z = \frac{\beta}{1-\beta\|Az\|_1} Az.$$

Now, from the same Theorem 2.2 in [4] we have that each solution of (8) belongs to the convex hull of the x_z's, hence also

$$x^3 \in \mathrm{Conv}\{\frac{\beta}{1-\beta\|Az\|_1} Az;\ z \in Z\}$$

which implies

$$x^1 = A^{-1}x^3 \in \mathrm{Conv}\{\frac{\beta}{1-\beta\|Az\|_1} z;\ z \in Z\}.$$

Thus for each $i \in \{1, \ldots, n\}$ we have

$$x_i^1 \le \frac{\beta}{1-\beta \max\{\|Az\|_1;\ z \in Z\}} = \frac{\beta}{1-\beta\|A\|_{\infty,1}}$$

and the upper bound is obviously achieved at some x_z which, due to (9) and (5), solves the equation

$$(A^{-1} - \beta z z^T)x_z = \beta z. \tag{10}$$

Hence for the $3n$-dimensional solution x of (6) we have

$$\overline{x}_i = \overline{x}_i^1 = \frac{\beta}{1-\beta\|A\|_{\infty,1}} \tag{11}$$

for each $i \in \{1, \ldots, n\}$ (cf. [5]).

5) Let $i \in \{1, \ldots, n\}$. Due to (11), (7) and (4) we have $\overline{x}_i \in (0, 1)$ and

$$\beta \ge \frac{1}{n(2n-1)+2} = \frac{1}{2n^2-n+2},$$

hence

$$\overline{x}_i \ge \frac{\beta}{1-\beta n} \ge \frac{\frac{1}{2n^2-n+2}}{1-\frac{n}{2n^2-n+2}} = \frac{1}{2n^2-2n+2}.$$

Since the real function $\frac{\xi^2}{1-\xi}$ is increasing in $(0, 1)$, we have

$$\frac{\overline{x}_i^2}{1-\overline{x}_i} \ge \frac{\frac{1}{(2n^2-2n+2)^2}}{1-\frac{1}{2n^2-2n+2}} = \frac{1}{(2n^2-2n+2)(2n^2-2n+1)} > \frac{1}{4n^4}.$$

Hence, if $\overline{y}_i$ satisfies (2), then

$$0 \le \overline{y}_i - \overline{x}_i < \frac{\overline{x}_i^2}{1-\overline{x}_i}$$

which implies

$$0 \le \overline{y}_i - \overline{x}_i < \overline{x}_i \overline{y}_i$$

and

$$0 \le \frac{1}{\overline{x}_i} - \frac{1}{\overline{y}_i} < 1. \qquad (12)$$

Now, from (11) we have

$$\|A\|_{\infty,1} = \frac{1}{\beta} - \frac{1}{\overline{x}_i}$$

and adding this to (12), we obtain

$$\|A\|_{\infty,1} \le \frac{1}{\beta} - \frac{1}{\overline{y}_i} < \|A\|_{\infty,1} + 1.$$

Since $\|A\|_{\infty,1}$ is integer for an MC-matrix A (due to (3)), the last result implies

$$\|A\|_{\infty,1} = [\frac{1}{\beta} - \frac{1}{\overline{y}_i}] = [\|A\|_s + 2 - \frac{1}{\overline{y}_i}].$$

Thus, if $\overline{y}_i$ satisfying (2) can be computed by a polynomial-time algorithm, then the same is true for $\|A\|_{\infty,1}$ and since computing $\|A\|_{\infty,1}$ is NP-hard for MC-matrices [3], P=NP follows. □

6 The symmetric case

Let $A^I = [A_c - \Delta, A_c + \Delta]$ be a symmetric interval matrix (i.e., the bounds $A_c - \Delta$ and $A_c + \Delta$ are symmetric) and let X^s be the set of solutions of (1) corresponding to systems with symmetric matrices only:

$$X^s = \{x;\ Ax = b \text{ for some } A \in A^I,\ b \in b^I,\ A \text{ symmetric}\}.$$

Again, $[\underline{y}, \overline{y}]$ is called an enclosure of X^s if $X^s \subseteq [\underline{y}, \overline{y}]$ holds. The narrowest enclosure is $[\underline{x}^s, \overline{x}^s]$, where

$$\underline{x}_i^s = \min_{X^s} x_i,$$
$$\overline{x}_i^s = \max_{X^s} x_i$$

for each i. We have an analogous result:

Theorem 2 *Suppose there exists a polynomial-time algorithm which for each strongly regular symmetric $n \times n$ interval matrix A^I and each b^I (both with rational bounds) computes a rational enclosure $[\underline{y}, \overline{y}]$ of X^s satisfying*

$$\overline{x}_i^s \le \overline{y}_i \le \overline{x}_i^s + \frac{1}{4n^4}$$

for each i. Then P=NP.

Proof. The system (6) constructed in the proof of Theorem 1 has a symmetric interval matrix A^I and each $\overline{x}_i,\ i = 1, \ldots, n$, is achieved at the solution of a system whose matrix is of the form

$$\begin{pmatrix} 0 & -I & 0 \\ -I & 0 & A^{-1} \\ 0 & A^{-1} & A^{-1} - \beta z z^T \end{pmatrix}$$

(eq. (10)), hence it is symmetric (since an MC-matrix A is symmetric). Thus we have

$$\overline{x}_i = \overline{x}_i^s$$

for $i = 1, \ldots, n$, and the proof of Theorem 1 applies to this case as well. □

References

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