1

LINEAR INTERVAL EQUATIONS: COMPUTING ENCLOSURES WITH BOUNDED RELATIVE OVERESTIMATION IS NP-HARD

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ABSTRACT

It is proved that if there exists a polynomial-time algorithm for enclosing solutions of linear interval equations with relative overestimation better than $\frac{4}{n^2}$ (where n is the number of equations), then P=NP. The result holds for the symmetric case as well.

1 INTRODUCTION

For a system of linear interval equations

$$A^I x = b^I \tag{1.1}$$

(A^I square), *enclosure* is defined as an interval vector $[\underline{y}, \overline{y}]$ satisfying

$$X \subseteq [\underline{y}, \overline{y}]$$

where X is the solution set:

$$X = \{x;\ Ax = b \text{ for some } A \in A^I,\ b \in b^I\}.$$

Various enclosure methods can be found in Alefeld and Herzberger [2] or Neumaier [7]. If A^I is regular, then there exists the narrowest (or: optimal) enclosure $[\underline{x}, \overline{x}]$ given by

$$\underline{x}_i = \min_X x_i,$$

$$\overline{x}_i = \max_X x_i$$

for each i. Computing $[\underline{x}, \overline{x}]$ was proved to be NP-hard (Rohn and Kreinovich [12]; also, Kreinovich, Lakeyev and Noskov [6] for the rectangular case). In this paper we show that the same is true for computing "sufficiently accurate" enclosures (Theorem 1), even in the symmetric case (Theorem 2).

2 THE RESULT

Theorem 1. *Suppose there exists a polynomial-time algorithm which for each strongly regular $n \times n$ interval matrix A^I and each b^I (both with rational bounds) computes a rational enclosure $[\underline{y}, \overline{y}]$ of X satisfying*

$$\left| \frac{\overline{y}_i - \overline{x}_i}{\overline{x}_i} \right| \leq \frac{4}{n^2} \tag{1.2}$$

for each i with $\overline{x}_i \neq 0$. Then P=NP.

Comments.

1) $A^I = [A_c - \Delta, A_c + \Delta]$ is called strongly regular if $\varrho(|A_c^{-1}|\Delta) < 1$ (a well-known sufficient regularity condition).

2) P and NP are the well-known complexity classes. The conjecture that P≠NP, although unproved, is widely believed to be true (Garey and Johnson [3]).

3) Hence, the problem of computing sufficiently accurate enclosures is very difficult: an existence of a polynomial-time algorithm yielding the accuracy (1.2) would imply polynomial-time solvability of all problems in the class NP. At the current stage of the complexity theory (conjecture P≠NP) this possibility cannot be excluded, but must be considered highly unlikely.

Proof. 1) Denote $e = (1, 1, \ldots, 1)^T \in R^n$ and $Z = \{z \in R^n;\ |z_i| = 1 \text{ for each } i\}$, so that Z is the set of all ± 1-vectors. We shall use matrix norms

$$\|M\|_s = e^T|M|e = \sum_i \sum_j |m_{ij}|$$

and

$$\|M\|_{\infty,1} = \max\{\|Mz\|_1;\ z \in Z\} \tag{1.3}$$

(where $\|x\|_1 = \sum_i |x_i|$; cf. [4]). $\lfloor \alpha \rfloor$ denotes the integer part of a real number α.

2) A real symmetric $n \times n$ matrix $M = (m_{ij})$ is called an MC-matrix if it is of the form

$$m_{ij} \begin{cases} = n & \text{if} \quad i = j \\ \in \{0, -1\} & \text{if} \quad i \neq j \end{cases}$$

$(i, j = 1, \ldots, n)$. For an MC-matrix M we obviously have

$$n \leq e^T M e \leq \|M\|_{\infty,1} \leq \|M\|_s \leq n(2n-1). \tag{1.4}$$

Also,

$$z_i (Mz)_i > 0 \tag{1.5}$$

holds for each $z \in Z$ and each $i \in \{1, \ldots, n\}$. We shall essentially use the fact that computing $\|M\|_{\infty,1}$ is NP-hard for MC-matrices [10, Corollary 7]. In the sequel we shall construct, for a given $n \times n$ MC-matrix M, a linear interval system with interval matrix of size $3n \times 3n$ such that if $\overline{y}_i$ satisfies (1.2), then

$$\|M\|_{\infty,1} = \lfloor \|M\|_s + 1 - \frac{1}{\overline{y}_i} \rfloor.$$

Hence, if such a $\overline{y}_i$ can be computed in polynomial time, then $\|M\|_{\infty,1}$ can also be computed in polynomial time and since this is an NP-hard problem, P=NP will follow.

3) For a given $n \times n$ MC-matrix M (which is diagonally dominant and therefore nonsingular), consider a linear interval system

$$A^I x = b^I \tag{1.6}$$

with $A^I = [A_c - \Delta, A_c + \Delta]$, $b^I = [b_c - \delta, b_c + \delta]$ given by

$$A_c = \begin{pmatrix} 0 & -I & 0 \\ -I & 0 & M^{-1} \\ 0 & M^{-1} & M^{-1} \end{pmatrix},$$

$$\Delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta e e^T \end{pmatrix}$$

(all the blocks are $n \times n$, I is the unit matrix),

$$b_c = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\delta = \begin{pmatrix} 0 \\ 0 \\ \beta e \end{pmatrix}$$

(all the blocks are $n \times 1$) and

$$\beta = \frac{1}{\|M\|_s + 1}. \tag{1.7}$$

We shall first prove that A^I is strongly regular. Since

$$A_c^{-1} = \begin{pmatrix} M^{-1} & -I & I \\ -I & 0 & 0 \\ I & 0 & M \end{pmatrix}$$

(as it can be easily verified), we have

$$|A_c^{-1}|\Delta = \begin{pmatrix} 0 & 0 & \beta ee^T \\ 0 & 0 & 0 \\ 0 & 0 & \beta|M|ee^T \end{pmatrix}.$$

This matrix has eigenvalues $\lambda = 0$ (multiple) and $\lambda = \beta\|M\|_s$. Hence $\varrho(|A_c^{-1}|\Delta) = \beta\|M\|_s < 1$ due to (1.7), and A^I is strongly regular.

4) For the linear interval system (1.6), consider a solution x of the linear system $\tilde{A}x = \tilde{b}$ for some $\tilde{A} \in A^I$, $\tilde{b} \in b^I$. If we decompose x as

$$x = \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{pmatrix},$$

then we have

$$\begin{aligned} x^{(1)} &= M^{-1}x^{(3)} \\ x^{(2)} &= 0 \\ M'x^{(3)} &= b' \end{aligned}$$

for some M', b' satisfying $|M^{-1} - M'| \le \beta ee^T$ and $|b'| \le \beta e$, hence $x^{(3)}$ is a solution of the linear interval system

$$[M^{-1} - \beta ee^T, M^{-1} + \beta ee^T]x' = [-\beta e, \beta e] \tag{1.8}$$

whose matrix is obviously again strongly regular. From [8, Thm. 2.2] we have that for each $z \in Z$ the equation

$$M^{-1}x = \beta(\|x\|_1 + 1)z \tag{1.9}$$

has a unique solution x_z. A direct substitution shows that the solution has the form

$$x_z = \frac{\beta}{1-\beta\|Mz\|_1} Mz.$$

Now, from the same Theorem 2.2 in [8] we have that each solution of (1.8) belongs to the convex hull of the x_z's, hence also

$$x^{(3)} \in \mathrm{Conv}\{\frac{\beta}{1-\beta\|Mz\|_1} Mz;\ z \in Z\}$$

which implies

$$x^{(1)} = M^{-1}x^{(3)} \in \mathrm{Conv}\{\frac{\beta}{1-\beta\|Mz\|_1} z;\ z \in Z\}.$$

Thus for each $i \in \{1,\dots,n\}$ we have

$$x_i^{(1)} \le \frac{\beta}{1-\beta\max\{\|Mz\|_1;\ z \in Z\}} = \frac{\beta}{1-\beta\|M\|_{\infty,1}}$$

and the upper bound is obviously achieved at some x_z which, due to (1.9) and (1.5), solves the equation

$$(M^{-1} - \beta zz^T)x_z = \beta z. \tag{1.10}$$

Hence for the $3n$-dimensional solution x of (1.6) we have

$$\overline{x}_i = \overline{x}_i^{(1)} = \frac{\beta}{1-\beta\|M\|_{\infty,1}} \tag{1.11}$$

for each $i \in \{1,\dots,n\}$ (cf. [12]).

5) Let $i \in \{1,\dots,n\}$. Due to (1.4), (1.7) and (1.11) we have

$$\beta \ge \frac{1}{n(2n-1)+1} = \frac{1}{2n^2-n+1}$$

and

$$\overline{x}_i \ge \frac{\beta}{1-\beta n} \ge \frac{\frac{1}{2n^2-n+1}}{1-\frac{n}{2n^2-n+1}} = \frac{1}{2n^2-2n+1} \ge \frac{1}{2n^2}.$$

Hence, if an enclosure $[\underline{y}, \overline{y}]$ of the solution set of (1.6) satisfies (1.2), then

$$0 \le \frac{\overline{y}_i - \overline{x}_i}{\overline{x}_i} \le \frac{4}{(3n)^2} < \frac{1}{2n^2} \le \overline{x}_i \le \overline{y}_i,$$

which implies

$$0 \le \frac{1}{\overline{x}_i} - \frac{1}{\overline{y}_i} < 1. \tag{1.12}$$

Now, from (1.11) we have

$$\|M\|_{\infty,1} = \frac{1}{\beta} - \frac{1}{\overline{x}_i}$$

and adding this to (1.12), we obtain

$$\|M\|_{\infty,1} \le \frac{1}{\beta} - \frac{1}{\overline{y}_i} < \|M\|_{\infty,1} + 1.$$

Since $\|M\|_{\infty,1}$ is integer for an MC-matrix M (due to (1.3)), the last result implies

$$\|M\|_{\infty,1} = \lfloor \frac{1}{\beta} - \frac{1}{\overline{y}_i} \rfloor = \lfloor \|M\|_s + 1 - \frac{1}{\overline{y}_i} \rfloor.$$

Thus, if $\overline{y}_i$ satisfying (1.2) can be computed by a polynomial-time algorithm, then the same is true for $\|M\|_{\infty,1}$ and since computing $\|M\|_{\infty,1}$ is NP-hard for MC-matrices [10], P=NP follows. $\square$

3 THE SYMMETRIC CASE

Let $A^I = [A_c - \Delta, A_c + \Delta]$ be a symmetric interval matrix (i.e., the bounds $A_c - \Delta$ and $A_c + \Delta$ are symmetric) and let X^s be the set of solutions of (1.1) corresponding to systems with symmetric matrices only:

$$X^s = \{x;\ Ax = b \text{ for some } A \in A^I,\ b \in b^I,\ A \text{ symmetric}\}.$$

Again, $[\underline{y}, \overline{y}]$ is called an enclosure of X^s if $X^s \subseteq [\underline{y}, \overline{y}]$ holds. Enclosure methods for the symmetric case were given by Jansson [5] and Alefeld and Mayer [1]. The narrowest enclosure is $[\underline{x}^s, \overline{x}^s]$, where

$$\underline{x}^s_i = \min_{X^s} x_i,$$

$$\overline{x}^s_i = \max_{X^s} x_i$$

for each i. We have an analogous result:

Theorem 2. *Suppose there exists a polynomial-time algorithm which for each strongly regular symmetric $n \times n$ interval matrix A^I and each b^I (both with* *rational bounds) computes a rational enclosure* $[\underline{y}, \overline{y}]$ *of* X^s *satisfying*

$$\left| \frac{\overline{y}_i - \overline{x}_i^s}{\overline{x}_i^s} \right| \leq \frac{4}{n^2}$$

for each i *with* $\overline{x}_i^s \neq 0$. *Then P=NP.*

Proof. The system (1.6) constructed in the proof of Theorem 1 has a symmetric interval matrix A^I and each $\overline{x}_i$, $i = 1, \ldots, n$, is achieved at the solution of a system whose matrix is of the form

$$\begin{pmatrix} 0 & -I & 0 \\ -I & 0 & M^{-1} \\ 0 & M^{-1} & M^{-1} - \beta z z^T \end{pmatrix}$$

(Eq. (1.10)), hence it is symmetric (since an MC-matrix M is symmetric). Thus we have

$$\overline{x}_i = \overline{x}_i^s \tag{1.13}$$

for $i = 1, \ldots, n$, and the proof of Theorem 1 applies to this case as well. □

In particular, (1.13) in view of (1.11) and [10, Corollary 7] implies that computing the narrowest enclosure $[\underline{x}^s, \overline{x}^s]$ is NP-hard. Hence, taking symmetry into account does not help to overcome the NP-hardness of computing the narrowest enclosure established in [12]. A related result [9] says that checking nonsingularity of all *symmetric* matrices contained in a symmetric interval matrix is NP-hard.

4 CONCLUDING REMARK

Small modifications in Eq. (1.7) and in the part 5) of the proof show that Theorem 1 also holds true if the relative overestimation bound (1.2) is replaced by the absolute overestimation bound

$$\overline{y}_i \leq \overline{x}_i + \frac{1}{4n^4} \tag{1.14}$$

for each i (see [11]). This form seems to be less appropriate than (1.2) since the term $\frac{1}{4n^4}$ in (1.14) is not related to the magnitude of $\overline{x}_i$.

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