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Complexity of Solving Linear Interval Equations

It is proved that computing enclosures of solutions of linear interval equations with overestimation bounded by a polynomial in the system size is NP-hard.

1. Introduction

Solving linear interval equations usually means computing enclosures. For a system of linear interval equations

$$A^I x = b^I \tag{1}$$

(A^I square), *enclosure* is defined as an interval vector $[\underline{y}, \overline{y}]$ satisfying

$$X \subseteq [\underline{y}, \overline{y}],$$

where X is the solution set:

$$X = \{x;\ Ax = b \text{ for some } A \in A^I,\ b \in b^I\}.$$

Various enclosure methods can be found in Alefeld and Herzberger [1]. If A^I is regular (i.e., each $A \in A^I$ is nonsingular), then there exists the narrowest enclosure $[\underline{x}, \overline{x}]$ given by

$$\underline{x}_i = \min_X x_i,$$

$$\overline{x}_i = \max_X x_i$$

for each i. Computing $[\underline{x}, \overline{x}]$ was proved to be NP-hard (Rohn and Kreinovich [7]; also, Kreinovich, Lakeyev and Noskov [4] for the rectangular case). In the main result of this paper we show that computing enclosures with overestimation bounded by a polynomial in the system size is NP-hard. The result holds true even for a very restricted class of systems (1) with $A^I = [A_c - \Delta, A_c + \Delta]$ having nondegenerate interval coefficients in one row only and satisfying $\varrho(|A_c^{-1}|\Delta) = 0$. Hence, the problem of computing sufficiently narrow enclosures turns out to be more difficult than previously believed.

2. Preliminaries

A real symmetric $n \times n$ matrix $A = (a_{ij})$ is called an MC-matrix [5] if it is of the form

$$a_{ij} \begin{cases} = n & \text{if} \quad i = j \\ \in \{0, -1\} & \text{if} \quad i \neq j \end{cases}$$

$(i, j = 1, \ldots, n)$. In the proof of the main theorem we shall essentially utilize the following result ([6], Corollary 7) concerning the norm

$$\|A\|_{\infty,1} = \max\{\|Ax\|_1;\ \|x\|_\infty = 1\}$$

(where $\|x\|_1 = \sum_i |x|_i$ and $\|x\|_\infty = \max_i |x_i|$; see Golub and van Loan [3], p. 15):

Proposition 1. *Computing $\|A\|_{\infty,1}$ is NP-hard for MC-matrices.*

Next we introduce a class of systems (1) of a special form. For each pair of rational numbers $\varepsilon > 0$, $\delta > 0$ we shall denote by $H_{\varepsilon\delta}$ the family of systems of linear interval equations

$$A^I x = b^I$$

with A^I of the form

$$A^I = \left(\begin{array}{cc} a & [-\varepsilon e^T, \varepsilon e^T] \\ 0 & A^{-1} \end{array} \right), \tag{2}$$

where a is a positive rational number, A is an $n \times n$ MC-matrix (n arbitrary, $n \geq 1$), $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ (i.e., A^I is $(n+1) \times (n+1)$), and

$$b^I = \left(\begin{array}{c} 0 \\ [-\delta e, \delta e] \end{array} \right) \tag{3}$$

is an $(n+1)$-dimensional interval vector. If we write (2) as

$$A^I = [A_c - \Delta, A_c + \Delta],$$

then

$$A_c = \left(\begin{array}{cc} a & 0^T \\ 0 & A^{-1} \end{array} \right)$$

is nonnegative symmetric positive definite [5], the radius matrix

$$\Delta = \left(\begin{array}{cc} 0 & \varepsilon e^T \\ 0 & 0 \end{array} \right)$$

has nonzero coefficients in the first row only, and

$$|A_c^{-1}|\Delta = \left(\begin{array}{cc} 0 & \frac{\varepsilon}{a} e^T \\ 0 & 0 \end{array} \right),$$

hence

$$\varrho(|A_c^{-1}|\Delta) = 0.$$

Thus the interval matrix (2) is strongly regular (i.e. $\varrho(|A_c^{-1}|\Delta) < 1$); problems with strongly regular interval matrices have been usually considered "tractable".

In order to be able to formulate a unifying complexity result, we introduce the following concept: *enclosure algorithm* is an algorithm which for each system $A^I x = b^I$ with rational data (and square A^I) in a finite number of steps either computes a rational enclosure, or fails (i.e., issues an error message). Failure of an enclosure algorithm may be caused by various reasons: 1) no enclosure exists since the solution set is unbounded (in case of a singular A^I), 2) the algorithm cannot be continued (e.g. in case of the interval Gaussian algorithm), 3) the algorithm works under some condition only (e.g., strong regularity), 4) a prescribed number of steps has been reached, etc.

3. Main result

Theorem 1. *If $P \neq NP$, then for each polynomial-time enclosure algorithm and each rational $\varepsilon > 0$, $\delta > 0$ either (i), or (ii) holds:*

(i) the algorithm fails for some system in $H_{\varepsilon\delta}$,

(ii) for each rational $\alpha > 0$ and each integer $k \geq 0$ there exists a system of size $n \geq 2$ in $H_{\varepsilon\delta}$ for which the enclosure $[\underline{y}, \overline{y}]$ computed by the algorithm satisfies

$$\underline{y}_1 \leq \underline{x}_1 - \alpha n^k < \overline{x}_1 + \alpha n^k \leq \overline{y}_1. \tag{4}$$

Remark 1. 1) P and NP are the well-known complexity classes. The conjecture that P≠NP, although unproved, is widely believed to be true (cf. Garey and Johnson [2]). 2) If the conjecture holds true, then each polynomial-time enclosure algorithm which works for at least one family $H_{\varepsilon\delta}$ may produce arbitrarily large overestimations (4); hence, no (even arbitrarily bad) accuracy can be guaranteed to be achievable by a polynomial-time enclosure algorithm.

Proof. Assume to the contrary that there exists a polynomial-time enclosure algorithm, rational numbers $\varepsilon > 0$, $\delta > 0$, $\alpha > 0$ and an integer $k \geq 0$ such that for each system in $H_{\varepsilon\delta}$ the algorithm computes an enclosure $[\underline{y}, \overline{y}]$ satisfying either

$$\underline{x}_1 - \alpha n^k < \underline{y}_1$$

or

$$\overline{y}_1 < \overline{x}_1 + \alpha n^k,$$

where n is the system size. Let A be an arbitrary MC-matrix of size m. Let us construct an $(m+1) \times (m+1)$ interval matrix

$$A^I = \begin{pmatrix} \frac{\varepsilon\delta}{\gamma} & [-\varepsilon e^T, \varepsilon e^T] \\ 0 & A^{-1} \end{pmatrix},$$

where

$$\gamma = \alpha(m+1)^k,$$

and an $(m+1)$-dimensional interval vector

$$b^I = \begin{pmatrix} 0 \\ [-\delta e, \delta e] \end{pmatrix},$$

and apply the algorithm to the system

$$A^I x = b^I \tag{5}$$

(which obviously belongs to $H_{\varepsilon\delta}$) to compute an enclosure $[\underline{y}, \overline{y}]$ which, according to the assumption, satisfies either

$$\underline{x}_1 - \gamma < \underline{y}_1 \tag{6}$$

or

$$\overline{y}_1 < \overline{x}_1 + \gamma. \tag{7}$$

This can be done in polynomial time. We shall prove that

$$\|A\|_{\infty,1} = \left[\frac{1}{\gamma}\min\{-\underline{y}_1, \overline{y}_1\}\right] \tag{8}$$

holds, where $[\ldots]$ denotes the integer part. Hence, $\|A\|_{\infty,1}$ can be computed in polynomial time; but since this is an NP-hard problem (Proposition 1), P=NP will follow. To prove (8), first observe that the system (5) can be written as

$$\frac{\varepsilon\delta}{\gamma}x_1 + [-\varepsilon e^T, \varepsilon e^T]x' = 0,$$

$$-\delta e \leq A^{-1}x' \leq \delta e,$$

where $x' = (x_2, \ldots, x_m)^T$. Hence

$$\begin{aligned} \overline{x}_1 &= \frac{\gamma}{\varepsilon\delta}\max\{\varepsilon e^T|x'|;\ -\delta e \leq A^{-1}x' \leq \delta e\} \\ &= \gamma\max\{\|x''\|_1;\ -e \leq A^{-1}x'' \leq e\} \\ &= \gamma\max\{\|Ax'''\|_1;\ -e \leq x''' \leq e\} \\ &= \gamma\max\{\|Ax'''\|_1;\ \|x'''\|_\infty = 1\} \\ &= \gamma\|A\|_{\infty,1} \end{aligned}$$

and in a quite similar way,

$$\underline{x}_1 = -\gamma\|A\|_{\infty,1}.$$

Hence from (6) and (7) we obtain that either

$$-\frac{1}{\gamma}\underline{y}_1 < \|A\|_{\infty,1} + 1$$

or

$$\frac{1}{\gamma}\overline{y}_1 < \|A\|_{\infty,1} + 1$$

holds, in both the cases

$$\frac{1}{\gamma}\min\{-\underline{y}_1, \overline{y}_1\} < \|A\|_{\infty,1} + 1. \tag{9}$$

But since $[\underline{y}_1, \overline{y}_1]$ encloses $[\underline{x}_1, \overline{x}_1]$, from $\underline{y}_1 \le \underline{x}_1$, $\overline{x}_1 \le \overline{y}_1$ we have

$$\|A\|_{\infty,1} \le \frac{1}{\gamma}\min\{-\underline{y}_1, \overline{y}_1\}$$

which together with (9) gives

$$\|A\|_{\infty,1} \le \frac{1}{\gamma}\min\{-\underline{y}_1, \overline{y}_1\} < \|A\|_{\infty,1} + 1. \tag{10}$$

However, the number

$$\|A\|_{\infty,1} = \max\{\|Ax\|_1;\ \|x\|_\infty = 1\} = \max\{\|Ax\|_1;\ x_j \in \{-1, 1\} \text{ for each } j\}$$

is integer for an MC-matrix A (which is integer by definition), hence from (10) we finally obtain that

$$\|A\|_{\infty,1} = \left[\frac{1}{\gamma}\min\{-\underline{y}_1, \overline{y}_1\}\right],$$

which is (8). Hence, $\|A\|_{\infty,1}$ can be computed in polynomial time for an MC-matrix A, which in view of Proposition 1 implies that P=NP. This concludes the proof by contradiction.

4. Application: interval Gaussian algorithm

For each rational $\varepsilon > 0, \delta > 0$, the interval Gaussian algorithm with partial pivoting [1] (which is polynomial-time) is performable for each system in $H_{\varepsilon\delta}$ since all the pivots are real and nonzero due to the special form of the system matrix (2). Hence, if P≠NP, then arbitrarily large overestimations (4) may occur for arbitrarily narrow system matrices (2) and arbitrarily narrow right-hand sides (3).

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5. References

1 ALEFELD, G., HERZBERGER, J.: Introduction to interval computations. Academic Press, New York 1983.

2 GAREY, M. E., JOHNSON, D. S.: Computers and intractability: a guide to the theory of NP-completeness. Freeman, San Francisco 1979.

3 GOLUB, G. H., VAN LOAN, C. F.: Matrix computations. The Johns Hopkins University Press, Baltimore 1983.

4 KREINOVICH, V., LAKEYEV, A. V., NOSKOV, S. I.: Optimal solution of interval linear systems is intractable (NP-hard). Interval Computations **1** (1993), 6–14.

5 ROHN, J.: Checking positive definiteness or stability of symmetric interval matrices is NP-hard. Commentat. Math. Univ. Carolinae **35** (1994), 795–797.

6 ROHN, J.: NP-hardness results for some linear and quadratic problems. Technical Report No. 619, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague 1995, 11 p.

7 ROHN, J., KREINOVICH, V.: Computing exact componentwise bounds on solutions of linear systems with interval data is NP-hard. SIAM J. Matr. Anal. Appl. **16** (1995), 415–420.

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