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Validated Solutions of Nonlinear Equations

We give an existence and uniqueness check for systems of nonlinear equations together with an iterative method which yields a validated enclosure of the solution at each iteration.

1. The result

We consider here a system of n nonlinear equations in n unknowns of the form

$$x = F(x) \tag{1}$$

over an n-dimensional hyperrectangle $[\hat{x}-d, \hat{x}+d] = \{x';\ \hat{x}-d \le x' \le \hat{x}+d\}$. Existence theorems for (1) were given by Miranda, Kantorovich, Smale and others (see the recent paper by Alefeld, Gienger and Potra [1] for a survey and a list of references). The following theorem gives an existence and uniqueness check and an iterative method which yields a validated enclosure of the solution of (1) at each iteration:

Theorem 1: *Let F map a hyperrectangle $[\hat{x}-d, \hat{x}+d] \subset \mathbb{R}^n$ into $\mathbb{R}^n$, and let there exist a nonnegative matrix H with the following properties:*

(i) $|F(x') - F(x'')| \le H|x' - x''|$ for each $x', x'' \in [\hat{x}-d, \hat{x}+d]$,

(ii) $|\hat{x} - F(\hat{x})| < (I - H)d$.

Then the equation (1) has a unique solution x^* in $[\hat{x}-d, \hat{x}+d]$, and the sequences $\{x_j\}_{j=0}^{\infty}$ and $\{d_j\}_{j=0}^{\infty}$ given by

$$x_{j+1} = F(x_j), \tag{2}$$

$$d_{j+1} = Hd_j \tag{3}$$

($j = 0, 1, \ldots$), $x_0 = \hat{x}$, $d_0 = d$, satisfy $x_j \to x^*$, $d_j \searrow 0$ and

$$x^* \in [x_j - d_j, x_j + d_j] \tag{4}$$

for each j. Moreover, the sequence

$$\{[x_j - d_j, x_j + d_j]\}_{j=0}^{\infty} \tag{5}$$

is nested.

Proof: 1) Since H is nonnegative, from (ii) we have $Hd < d$ and $d > 0$, hence $\varrho(H) < 1$, $(I-H)^{-1} \ge 0$, and $H^j \to 0$ (Neumaier [2], sect. 3.2), thus also $d_j = H^j d \to 0$.

2) We shall prove by induction that

$$x_j \in [\hat{x}-d, \hat{x}+d] \tag{6}$$

for each j (hence the sequence $\{x_j\}$ is well defined by (2)) and

$$|x_j - x_{j+1}| \le d_j - d_{j+1} \tag{7}$$

for each j. For $j = 0$ we obviously have $x_0 = \hat{x} \in [\hat{x}-d, \hat{x}+d]$ and $|x_0 - x_1| = |\hat{x} - F(\hat{x})| < (I-H)d = d_0 - d_1$ due to (ii). Assume that (6) and (7) hold for $j = 0, \ldots, k-1$. Then $|\hat{x} - x_k| = |\sum_{j=0}^{k-1}(x_j - x_{j+1})| \le \sum_{j=0}^{k-1}|x_j - x_{j+1}| \le \sum_{j=0}^{k-1}(d_j - d_{j+1}) = d_0 - d_k \le d_0 = d$ (since $d_k \ge 0$ due to (3)), hence $x_k \in [\hat{x}-d, \hat{x}+d]$. Next, by (i) and by (7) for $j = k-1$ we obtain $|x_k - x_{k+1}| = |F(x_{k-1}) - F(x_k)| \le H|x_{k-1} - x_k| \le H(d_{k-1} - d_k) = d_k - d_{k+1}$, which concludes the inductive proof of (6) and (7). Since $\{d_j\}$ is decreasing by (7) and $d_j \to 0$ (cf. 1)), we have that $d_j \searrow 0$.

3) For each $j \geq 0$ and $m \geq 1$, from (7) we have $|x_j - x_{j+m}| \leq \sum_{k=j}^{j+m-1} |x_k - x_{k+1}| \leq \sum_{k=j}^{j+m-1}(d_k - d_{k+1}) = d_j - d_{j+m}$, hence

$$|x_j - x_{j+m}| \leq d_j - d_{j+m}. \tag{8}$$

Since $\{d_j\}$ is convergent, for each positive vector $\varepsilon > 0$ there exists a $j \geq 0$ such that $|d_j - d_{j+m}| = d_j - d_{j+m} < \varepsilon$ for each $m \geq 1$. Then (8) gives that $|x_j - x_{j+m}| < \varepsilon$, hence $\{x_j\}$ is Cauchian, so that $x_j \to x^*$, and (6) implies that $x^* \in [\hat{x} - d, \hat{x} + d]$.

4) Since F is continuous in $[\hat{x} - d, \hat{x} + d]$ due to (i), taking $j \to \infty$ in (2), we obtain $x^* = F(x^*)$. Let $\tilde{x}$ be any other solution to (1) in $[\hat{x} - d, \hat{x} + d]$. Then from $|\tilde{x} - x^*| = |F(\tilde{x}) - F(x^*)| \leq H|\tilde{x} - x^*|$ (due to (i)) we have $(I - H)|\tilde{x} - x^*| \leq 0$, and premultiplying this inequality by the nonnegative matrix $(I - H)^{-1}$ (cf. 1)) yields $|\tilde{x} - x^*| \leq 0$, hence $\tilde{x} = x^*$. Thus x^* is the unique solution of (1) in $[\hat{x} - d, \hat{x} + d]$.

5) For each $j \geq 0$, taking $m \to \infty$ in (8), we obtain $|x_j - x^*| \leq d_j$, hence $x^* \in [x_j - d_j, x_j + d_j]$, which proves (4). From (7) it follows that $x_j - d_j \leq x_{j+1} - d_{j+1}$ and $x_{j+1} + d_{j+1} \leq x_j + d_j$ for each j, hence the sequence of hyperrectangles (5) is nested. This completes the proof. ∎

In practice, the original problem is usually given in the form $f(x) = 0$, which is brought to the form (1) by employing a mapping $F(x) = x - Rf(x)$, where R is some nonsingular matrix. The second theorem shows that under mild assumptions the conditions (i), (ii) imposed on F in Theorem 1 are satisfied in a neighbourhood of the solution x^*. The Jacobian matrices of f and F are denoted by J_f, J_F, respectively, and ϱ is the spectral radius:

Theorem 2: *Let $f(x^*) = 0$ and let f have continuous partial derivatives in a neighbourhood of x^*. Then for each $n \times n$ matrix R satisfying*

$$\varrho(|I - RJ_f(x^*)|) < 1 \tag{9}$$

there exists a $d > 0$ such that the mapping

$$F(x) = x - Rf(x) \tag{10}$$

satisfies the assumptions (i), (ii) of Theorem 1 in $[x^* - d, x^* + d]$.

Proof: First, the mapping F given by (10) obviously satisfies $J_F(x) = I - RJ_f(x)$, hence (9) gives $\varrho(|J_F(x^*)|) < 1$. Let $\tilde{d} > 0$ be such that f has continuous partial derivatives in $[x^* - \tilde{d}, x^* + \tilde{d}]$. For each d' satisfying $0 \leq d' \leq \tilde{d}$ define a matrix $H(d') = (h_{ij}(d'))$ by

$$h_{ij}(d') = \max\left\{\left|\frac{\partial F_i}{\partial x_j}(x)\right|; x \in [x^* - d', x^* + d']\right\} \tag{11}$$

$(i, j = 1, \ldots, n)$. Then $H(0) = |J_F(x^*)|$ and $\varrho(H(0)) = \varrho(|J_F(x^*)|) < 1$, hence in view of continuity of the spectral radius there exists a $d' > 0$ with $d' \leq \tilde{d}$ such that $\varrho(H(d')) < 1$. Since $H(d')$ is nonnegative, there exists a vector $d'' > 0$ satisfying $H(d')d'' < d''$ (Neumaier [2]). Take a sufficiently small real number $\alpha > 0$ such that $\alpha d'' \leq d'$, and put $d = \alpha d''$, $H = H(d')$. Then $d \leq d'$ implies $H(d) \leq H$ by (11), and from $H(d')d'' < d''$ we have $Hd < d$. Hence for each $x', x'' \in [x^* - d, x^* + d]$ we have by the mean-value theorem and by (11) that $|F(x') - F(x'')| \leq H(d)|x' - x''| \leq H|x' - x''|$ and $|x^* - F(x^*)| = 0 < (I - H)d$ hold, hence the assumptions (i) and (ii) of Theorem 1 are satisfied in $[x^* - d, x^* + d]$. ∎

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2. References

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