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Linear Programming with Inexact Data is NP-Hard

We prove that the problem of checking existence of optimal solutions to all linear programming problems whose data range in prescribed intervals is NP-hard.

1. The result

Consider a family of linear programming (LP) problems

$$\min\{c^T x;\ Ax = b,\ x \geq 0\} \tag{1}$$

for all data satisfying

$$A \in A^I,\ b \in b^I,\ c \in c^I, \tag{2}$$

where $A^I = \{A;\ \underline{A} \leq A \leq \overline{A}\}$ is an $m \times n$ interval matrix, $m \leq n$, and $b^I = \{b;\ \underline{b} \leq b \leq \overline{b}\}$, $c^I = \{c;\ \underline{c} \leq c \leq \overline{c}\}$ are interval vectors of dimensions m and n, respectively (the inequalities are understood componentwise). The family (1), (2) may be interpreted as a linear programming problem with inexact data, or as a fully parametrized parametric linear programming problem.

The problem of existence of optimal solutions to all linear programming problems in the family (1), (2) was addressed in [5]. There it was proved that each LP problem (1) with data satisfying (2) has an optimal solution if and only if the LP problem $\min\{\underline{c}^T x;\ \underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0\}$ has an optimal solution, and each of the 2^m systems $Ax = b$ whose each row is either of the form $(\underline{A}x)_i = \overline{b}_i$ or of the form $(\overline{A}x)_i = \underline{b}_i$ $(i = 1, \ldots, m)$ has a nonnegative solution. Hence, we have a finitely verifiable necessary and sufficient condition, but the number of systems to be checked for nonnegative solvability is exponential in m.

In the main result of this paper we show that the problem in question is NP-hard. Hence, unless the famous conjecture "P≠NP" (see GAREY AND JOHNSON [1]) is false, there does not exist a polynomial-time algorithm for checking existence of optimal solutions to all LP problems (1), (2). The proof given below shows that even checking *feasibility* of all LP problems in the family (1), (2) is NP-hard.

Theorem 1. *The following decision problem is NP-hard:*
Instance. A^I, b^I, c^I *(with rational bounds).*
Question. *Does each LP problem (1) with data satisfying (2) have an optimal solution?*

Proof. 0) For the purpose of the proof, let us introduce $A_c = \frac{1}{2}(\underline{A} + \overline{A})$, $\Delta = \frac{1}{2}(\overline{A} - \underline{A})$, $b_c = \frac{1}{2}(\underline{b} + \overline{b})$ and $\delta = \frac{1}{2}(\overline{b} - \underline{b})$, so that $A^I = [A_c - \Delta, A_c + \Delta]$ and $b^I = [b_c - \delta, b_c + \delta]$. The proof goes through several steps.

1) First we prove that each system

$$Ax = b,\ x \geq 0 \tag{3}$$

with data satisfying

$$A \in A^I,\ b \in b^I \tag{4}$$

has a solution if and only if

$$(\forall y)(A_c^T y + \Delta^T |y| \geq 0 \Rightarrow b_c^T y - \delta^T |y| \geq 0) \tag{5}$$

holds. "Only if": Let each system (3) with data (4) have a solution, and let $A_c^T y + \Delta^T |y| \geq 0$ for some $y \in \mathbb{R}^m$. Define a diagonal matrix T by $T_{ii} = 1$ if $y_i \geq 0$, $T_{ii} = -1$ if $y_i < 0$, and $T_{ij} = 0$ if $i \neq j$ $(i, j = 1, \ldots, m)$, then $|y| = Ty$. Consider now the system

$$(A_c + T\Delta)x = b_c - T\delta,\ x \geq 0. \tag{6}$$

Since $A_c + T\Delta \in A^I$ and $b_c - T\delta \in b^I$, the system (6) has a solution according to the assumption, and $(A_c + T\Delta)^T y = A_c^T y + \Delta^T |y| \geq 0$, hence FARKAS lemma applied to (6) gives that $b_c^T y - \delta^T |y| = (b_c - T\delta)^T y \geq 0$, which proves (5). "If": Assuming that (5) holds, consider a system (3) with data satisfying (4). Let $A^T y \geq 0$ for some y; then $A_c^T y + \Delta^T |y| \geq (A_c + A - A_c)^T y = A^T y \geq 0$, hence (5) gives that $b^T y = (b_c + b - b_c)^T y \geq b_c^T y - \delta^T |y| \geq 0$. Thus we have proved that for each y, $A^T y \geq 0$ implies $b^T y \geq 0$, and FARKAS lemma proves the existence of a solution to (3).

2) For a given *square* $m \times m$ interval matrix $A_0^I = [A_c^0 - \Delta^0, A_c^0 + \Delta^0]$, construct an $m \times 2m$ interval matrix

$$A^I = [A_c - \Delta, A_c + \Delta] \tag{7}$$

with

$$A_c = (A_c^{0T}, -A_c^{0T}),\ \Delta = (\Delta^{0T}, \Delta^{0T}), \tag{8}$$

and interval vectors

$$b^I = [-e, e],\ c^I = [e, e], \tag{9}$$

where $e = (1, \ldots, 1)^T$. We shall prove that A_0^I is regular (i.e., each $A \in A_0^I$ is nonsingular) if and only if each LP problem (1) with data satisfying (2) (A^I, b^I, c^I given by (7)–(9)) has an optimal solution. In fact, since the objective $e^T x$ is bounded from below, a problem (1) has an optimal solution if and only if it is feasible. Hence, according to part 1), Eq. (5), some problem (1) with data (2) does *not* have an optimal solution if and only if there exists a vector y satisfying $\begin{pmatrix} A_c^0 \\ -A_c^0 \end{pmatrix} y + \begin{pmatrix} \Delta^0 \\ \Delta^0 \end{pmatrix} |y| \geq 0$ and $e^T |y| > 0$, which is equivalent to

$$|A_c^0 y| \leq \Delta^0 |y|,\ y \neq 0. \tag{10}$$

Then the OETTLI–PRAGER theorem [3] gives that (10) is equivalent to existence of a singular matrix in $A_0^I = [A_c^0 - \Delta^0, A_c^0 + \Delta^0]$. This proves the assertion.

3) Given a square $m \times m$ interval matrix A_0^I, construct an $m \times 2m$ interval matrix A^I and interval vectors b^I, c^I by (7)–(9). According to part 2), checking regularity of A_0^I can be reduced in polynomial time to checking optimality of all problems (1), (2). But since the problem of checking regularity of interval matrices is NP-hard (POLJAK AND ROHN [4], Theorem 2.8), the problem of checking whether each LP problem (1) with data satisfying (2) has an optimal solution is NP-hard as well. ■

2. Concluding remarks

KHACHIYAN [2] proved that an LP problem (1) can be solved in polynomial time. The above result shows that this nice property is lost when inexact data are present. Nevertheless, the worst–case–type result of Theorem 1 does not preclude efficient solvability of many practical examples. The criterion from [5] quoted in the introduction requires solving one LP problem and checking nonnegative solvability of 2^p systems of linear equations, where p is the number of rows i having at least one inexact coefficient (i.e., either $\underline{b}_i < \overline{b}_i$, or $\underline{A}_{ij} < \overline{A}_{ij}$ for some j). Thus the criterion can be efficiently applied to practical examples with small values of p.

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3. References

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