The Conjecture "P≠NP" and Overestimation in Bounding Solutions of Perturbed Linear Equations*

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Abstract

It is proved that a classical bound on solutions of perturbed systems of linear equations may yield arbitrarily large polynomial overestimations for arbitrarily narrow perturbations provided the conjecture "P≠NP" is true.

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1 Introduction

For a system of linear equations

$$Ax = b \tag{1}$$

with an $n \times n$ nonsingular matrix A, consider a family of perturbed systems

$$A'x' = b' \tag{2}$$

with data satisfying

$$|A' - A| \le \Delta \tag{3}$$

and

$$|b' - b| \le \delta, \tag{4}$$

where $\Delta \ge 0$ and $\delta \ge 0$ are an $n \times n$ perturbation matrix and a perturbation n-vector, respectively, and the inequalities are understood componentwise. The classical numerical argument using Neumann series shows that if the spectral condition

$$\varrho(|A^{-1}|\Delta) < 1 \tag{5}$$

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holds, then each A' satisfying (3) is nonsingular and the solution of each system (2) with data (3), (4) satisfies

$$|x' - x| \leq d, \tag{6}$$

where

$$d = (I - |A^{-1}|\Delta)^{-1}|A^{-1}|(\Delta|x| + \delta) \tag{7}$$

and I is the unit matrix (see Skeel [8] or Rump [6]). To keep the paper self–contained, we give here another simple proof of this result: for the solutions x, x' of (1), (2) under (3), (4) we have

$$\begin{aligned} |x' - x| &= |A^{-1}A(x' - x)| \leq |A^{-1}| \cdot |(A - A')(x' - x) + (A - A')x + b' - b| \\ &\leq |A^{-1}|(\Delta|x' - x| + \Delta|x| + \delta), \end{aligned}$$

hence

$$(I - |A^{-1}|\Delta)|x' - x| \leq |A^{-1}|(\Delta|x| + \delta)$$

and premultiplying this inequality by $(I - |A^{-1}|\Delta)^{-1}$, which is nonnegative in view of (5), we obtain (6), where d is given by (7).

The quality of the estimation (6) has been paid little attention in the literature. Obviously, the bound d is exact if $\Delta = 0$. In fact, in this case, for each $i \in \{1, \ldots, n\}$, if we take $b'_j = b_j + \delta_j$ if $(A^{-1})_{ij} \geq 0$ and $b'_j = b_j - \delta_j$ otherwise, then b' satisfies (4) and for the solution x' of $Ax' = b'$ we have

$$|x'_i - x_i| = \sum_j |(A^{-1})_{ij}|\delta_j = d_i,$$

hence the bound is achieved. However, this argument fails in the case $\Delta \neq 0$. In this paper we show that the famous conjecture "P≠NP" (see Garey and Johnson [1] for details) shreds a surprising light on this problem: in the main result to follow we show that if the conjecture is true, then the formula (6) may yield an arbitrarily large polynomial overestimation for arbitrarily narrow perturbations Δ, δ. Hence, the conjecture penetrates the area of numerical linear algebra as well.

2 Main result

We shall use the subordinate matrix norm

$$\|\Delta\|_m = \max_{i,j} |\Delta_{ij}|$$

and the vector norm

$$\|\delta\|_\infty = \max_i |\delta_i|.$$

Our main result is formulated as follows:

Theorem 1 *If P≠NP, then for each rational $\varepsilon > 0$, $\eta > 0$, $\alpha > 0$ and for each integer $k \geq 0$ there exist $n \times n$ matrices A, $\Delta \geq 0$ and n-vectors b, $\delta \geq 0$ for some $n \geq 2$ such that*

$$\varrho(|A^{-1}|\Delta) = 0 \tag{8}$$

$$\|\Delta\|_m = \varepsilon \tag{9}$$

$$\|\delta\|_\infty = \eta \tag{10}$$

hold and the solution x' of each system (2) with data (3), (4) satisfies

$$|x'_1 - x_1| + \alpha n^k \leq d_1, \tag{11}$$

where x is the solution of (1) and d is given by (7).

Proof. Assume to the contrary that it is not so, so that there exist rational numbers $\varepsilon > 0$, $\eta > 0$, $\alpha > 0$ and an integer $k \geq 0$ such that for each $n \geq 2$ and all $n \times n$ matrices A, $\Delta \geq 0$ and all n-vectors b, $\delta \geq 0$ satisfying (8)–(10) we have

$$|x'_1 - x_1| + \alpha n^k > d_1 \tag{12}$$

for the solution x' of some system (2) with data (3), (4).

Take an arbitrary $m \times m$ MC-matrix $\tilde{A}$, $m \geq 1$, i.e. a matrix $\tilde{A}$ satisfying $\tilde{A}_{ii} = m$ and $\tilde{A}_{ij} \in \{0, -1\}$ if $i \neq j$ $(i, j = 1, \ldots, m)$; $\tilde{A}$ is nonsingular (cf. [4]). Let us define

$$A = \begin{pmatrix} \frac{\varepsilon\eta}{\gamma} & 0^T \\ 0 & \tilde{A}^{-1} \end{pmatrix}, \tag{13}$$

$$\Delta = \begin{pmatrix} 0 & \varepsilon e^T \\ 0 & 0 \end{pmatrix}, \tag{14}$$

where $\gamma = \alpha(m+1)^k$ and $e = (1, \ldots, 1)^T \in \mathbb{R}^m$ (hence A and Δ are of size $(m+1) \times (m+1)$), and let

$$b = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{15}$$

and

$$\delta = \begin{pmatrix} 0 \\ \eta e \end{pmatrix} \tag{16}$$

be $(m+1)$-dimensional vectors. Then

$$|A^{-1}|\Delta = \begin{pmatrix} 0 & \frac{\gamma}{\eta} e^T \\ 0 & 0 \end{pmatrix},$$

hence (8), (9) and (10) hold, the solution of (1) is $x = 0$ and for

$$\overline{x}_1 := \max\{x'_1;\ x' \text{ solves (2) under (3), (4)}\}$$

we have (if we denote $\tilde{x} = (x_2, x_3, \ldots, x_m)^T$) that

$$\begin{aligned}\overline{x}_1 &= \frac{\gamma}{\varepsilon\eta}\max\{\varepsilon e^T|\tilde{x}|;\ -\eta e \le \tilde{A}^{-1}\tilde{x} \le \eta e\} \\ &= \gamma\max\{\|\tilde{A}x\|_1;\ x_j \in \{-1,1\} \text{ for each } j\} \\ &= \gamma\|\tilde{A}\|_{\infty,1}\end{aligned}$$

(see Golub and van Loan [2] for definition of $\|\tilde{A}\|_{\infty,1}$), and in a similar way for

$$\underline{x}_1 := \min\{x'_1;\ x' \text{ solves (2) under (3), (4)}\}$$

we obtain

$$\underline{x}_1 = -\gamma\|\tilde{A}\|_{\infty,1}.$$

Let us now compute d by (7). Then in view of (12) we have (since $x = 0$) that

$$\gamma\|\tilde{A}\|_{\infty,1} \ge |x'_1| > d_1 - \alpha(m+1)^k = d_1 - \gamma,$$

hence

$$d_1 < \gamma(\|\tilde{A}\|_{\infty,1} + 1). \tag{17}$$

But in view of (6) and of $x = 0$ we also have

$$\gamma\|\tilde{A}\|_{\infty,1} = \overline{x}_1 \le d_1, \tag{18}$$

hence (17) and (18) give

$$\|\tilde{A}\|_{\infty,1} \le \frac{d_1}{\gamma} < \|\tilde{A}\|_{\infty,1} + 1. \tag{19}$$

Since the MC-matrix $\tilde{A}$ is integer by definition, the number

$$\|\tilde{A}\|_{\infty,1} = \max\{\|\tilde{A}x\|_1;\ x_j \in \{-1,1\} \text{ for each } j\}$$

is also integer, hence from (19) we finally obtain

$$\|\tilde{A}\|_{\infty,1} = \left[\frac{d_1}{\gamma}\right], \tag{20}$$

where $[\ldots]$ denotes the integer part.

Summing up, we have proved the following: given an MC-matrix $\tilde{A}$, if we construct A, Δ, b and δ by (13)–(16) and then compute d by (7), then (20) holds. Since all these computations can be done in polynomial time (Schrijver [7]), we have a polynomial-time algorithm for computing $\|\tilde{A}\|_{\infty,1}$ for an MC-matrix $\tilde{A}$. However, computing $\|\tilde{A}\|_{\infty,1}$ was proved to be NP-hard for MC-matrices $\tilde{A}$ ([5], Corollary 7, which is a simple consequence of Theorem 2.6 in [3]). Hence, an existence of a polynomial-time algorithm for solving an NP-hard problem implies P=NP, which contradicts our assumption.

■

3 Concluding remarks

We have proved that if P≠NP, then for arbitrarily narrow perturbations (9), (10) the formula (7) may yield a catastrophic overestimation (11). This, of course, is a worst-case-type result. The conjecture "P≠NP" has not been proved to date, but it is widely believed to be true (Garey and Johnson [1]). In any case, we can see that the conjecture is closely related to one of the basic problems in numerical linear algebra; if the assertion concerning the overestimation (11) is not true, then a simple algorithm based on formulae (13), (14), (15), (16), (7) and (20) gives a polynomial-time algorithm for solving an NP-hard problem, thereby also solving in polynomial time all the problems in the class NP.

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