SUFFICIENT CONDITIONS FOR REGULARITY AND SINGULARITY OF INTERVAL MATRICES

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Abstract. Several verifiable sufficient conditions for regularity and singularity of interval matrices are given. As an application, a verifiable sufficient condition is derived for an interval matrix to have all eigenvalues real.

Key words. Interval matrix, regularity, singularity, inverse matrix, eigenvalue, positive definiteness

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1. Introduction.

As is well known, an interval matrix

$$A^I = [\underline{A}, \overline{A}] = \{A;\ \underline{A} \le A \le \overline{A}\}$$

(where $\underline{A}$ and $\overline{A}$ are $n \times n$ matrices and the inequalities are understood componentwise) is called regular if each $A \in A^I$ is nonsingular, and is said to be singular otherwise (i.e., if it contains a singular matrix). Regularity of interval matrices plays an important role in the theory of linear interval equations (cf. Neumaier [13]), but it is also important in another respects since several frequently used properties of interval matrices (as positive definiteness, P-property, stability and Schur stability) may be verified via checking regularity (see Rohn and Rex [24], Rohn [22]).

The problem of checking regularity of interval matrices has been proved to be NP-hard (Poljak and Rohn [15], [16], see also Nemirovskii [10]). In its most recent version [23], the result says that for each rational $\varepsilon > 0$ checking regularity is NP-hard in the class of interval matrices of the form

$$[A - \varepsilon E, A + \varepsilon E],$$

where A is a nonnegative symmetric positive definite rational matrix and E is the matrix of all ones.

In view of this NP-hardness result and of the current status of the complexity theory (the conjecture "P≠NP", cf. Garey and Johnson [3]), no easily performable (i.e., polynomial-time) algorithms for checking regularity of interval matrices may be expected to exist. In practical computations we must therefore resort to verifiable sufficient conditions for both regularity and singularity of interval matrices. In order to cover a possibly wide class of interval matrices, it is recommendable to have more such conditions at one's disposal since some sufficient conditions may be better suited for specific classes of interval matrices than the other ones. Such a situation is well known for the problem of stability of interval matrices which is also NP-hard (Nemirovskii [10], Rohn [23]), where a number of sufficient conditions of different types are known, see the survey paper by Mansour [9].

The purpose of this paper is three-fold. First, we give three sufficient regularity conditions and three sufficient singularity conditions, grouped into pairs according to

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their form (conditions using midpoint inverse, conditions using eigenvalues and those based on checking positive definiteness). Two of them (Theorems 3.1 and 4.1) have been already known, the others are new (Theorems 3.3, 4.2, 5.1 and 5.2). Second, we show that all these verifiable sufficient conditions can be derived in a rather uniform way from two necessary and sufficient conditions that themselves are not of practical use since they require a number of arithmetic operations which is exponential in the matrix size n. Third, as an application of the previous results we give in Theorem 6.1 a verifiable sufficient condition for an interval matrix to have all eigenvalues real.

We shall use the following notations. The absolute value of a matrix $A = (a_{ij})$ is denoted by $|A| = (|a_{ij}|)$; the same notation applies to vectors as well. $\varrho(A)$ is the spectral radius of A, and $\lambda_{\min}(A)$, $\lambda_{\max}(A)$ stand for the minimal and maximal eigenvalue of a symmetric matrix A, respectively. As is well known, $\lambda_{\min}(A) = \min_{\|x\|_2=1} x^T Ax$ and $\lambda_{\max}(A) = \max_{\|x\|_2=1} x^T Ax$ hold for a symmetric matrix A, see Golub and van Loan [4]. I denotes the unit matrix.

2. Necessary and sufficient conditions. For an interval matrix

$$A^I = [\underline{A}, \overline{A}], \tag{2.1}$$

let us introduce

$$A_c = \tfrac{1}{2}(\underline{A} + \overline{A})$$

(the midpoint matrix) and

$$\Delta = \tfrac{1}{2}(\overline{A} - \underline{A})$$

(the radius matrix). Then we can write (2.1) as

$$A^I = [A_c - \Delta, A_c + \Delta], \tag{2.2}$$

which form is better suited for formulations of the subsequent conditions.

The first known necessary and sufficient condition for singularity of interval matrices is due to Oettli and Prager. In fact, the formulation given below cannot be found explicitly in their original paper [14], but it follows easily from the basic theorem on linear interval equations given there when applied to systems with zero right-hand sides (see Neumaier [13] or Rohn [20]).

THEOREM 2.1. (Oettli and Prager [14]) *An interval matrix (2.2) is singular if and only if the inequality*

$$|A_c x| \le \Delta |x| \tag{2.3}$$

has a nontrivial solution.

The proof, given e.g. in [20], is constructive: if (2.3) holds for some $x \neq 0$, then for the vectors $y, z \in \mathbb{R}^n$ defined by

$$y_i = \begin{cases} (A_c x)_i/(\Delta|x|)_i & \text{if } (\Delta|x|)_i \neq 0, \\ 1 & \text{if } (\Delta|x|)_i = 0 \end{cases}$$

and

$$z_j = \begin{cases} 1 & \text{if } x_j \ge 0, \\ -1 & \text{if } x_j < 0 \end{cases}$$

$(i, j = 1, \ldots, n)$, the matrix A given by

$$A_{ij} = (A_c)_{ij} - y_i z_j \Delta_{ij}$$

$(i, j = 1, \ldots, n)$ belongs to A^I and is singular (since $Ax = 0$). Hence, we can construct a singular matrix in A^I if we know a nontrivial solution to (2.3). However, in view of the NP-hardness result, such a solution is not to be found easily.

The following necessary and sufficient regularity condition employs again the inequality of the form (2.3), with the "$\leq$" sign being converted to "$>$". We emphasize that the strict inequality is again meant componentwise.

THEOREM 2.2. (Rohn [21]) *An interval matrix (2.2) is regular if and only if for each orthant $\mathcal{O}$ of $\mathbb{R}^n$ there exists a solution of the inequality*

$$|A_c x| > \Delta |x| \tag{2.4}$$

satisfying $A_c x \in \mathcal{O}$.

Unlike the previous theorem, the proof of this result is more involved and employs some nontrivial facts concerning the linear complementarity problem, P-matrices and solvability of linear equations. Again, Theorem 2.2 is of little practical use since it requires a proof of existence of 2^n solutions of the inequality (2.4). However, Theorems 2.1 and 2.2 form a basis for deriving some verifiable sufficient conditions for regularity and singularity of interval matrices that will be given in the three subsequent sections.

3. Sufficient conditions using the midpoint inverse. Two known sufficient conditions (given below as Corollary 3.2 and Corollary 3.4) use the midpoint inverse A_c^{-1} in their formulations. Since using the inverse matrix computed in a finite precision arithmetic may affect validity of these conditions, it is advantageous to formulate them in terms of an approximate inverse R instead of the exact inverse A_c^{-1}. The first such formulation appeared, although implicitly, in Rump's paper [25]. We reprove the condition here using another idea based on Theorem 2.1:

THEOREM 3.1. *Let R be an arbitrary matrix such that*

$$\varrho(|I - RA_c| + |R|\Delta) < 1 \tag{3.1}$$

holds. Then $[A_c - \Delta, A_c + \Delta]$ is regular.

Proof. Assume to the contrary that $[A_c - \Delta, A_c + \Delta]$ is singular, so that by Theorem 2.1 there exists an $x \neq 0$ satisfying $|A_c x| \leq \Delta |x|$. Then we have

$$\begin{aligned} |x| &= |(I - RA_c)x + RA_c x| \leq |I - RA_c| \cdot |x| + |R| \cdot |A_c x| \\ &\leq (|I - RA_c| + |R|\Delta)|x|, \end{aligned}$$

hence

$$1 \leq \varrho(|I - RA_c| + |R|\Delta)$$

by Perron-Frobenius theorem (see Neumaier [13], Corollary 3.2.3), a contradiction. ∎

If we take $R = A_c^{-1}$, we immediately obtain the following result:

COROLLARY 3.2. *Let A_c be nonsingular and*

$$\varrho(|A_c^{-1}|\Delta) < 1 \tag{3.2}$$

hold. Then $[A_c - \Delta, A_c + \Delta]$ is regular.

The condition (3.2) was first published by Beeck [2], although allegedly (Neumaier [12]) its priority is due to Ris who had proved it earlier in his unpublished PhD thesis [19].

In his recent papers [28], [26], Rump proved that each regular $n \times n$ interval matrix $[A_c - \Delta, A_c + \Delta]$ satisfies

$$\varrho(|A_c^{-1}|\Delta) < (3 + 2\sqrt{2})n,$$

and that for each $n \geq 1$ there exists a regular $n \times n$ interval matrix such that

$$\varrho(|A_c^{-1}|\Delta) > n - 1.$$

These facts help to clarify the strength of the sufficient condition (3.2).

Since (3.2) is a special case of (3.1) for $R = A_c^{-1}$, it was believed for some time that (3.1) is more general than (3.2). Rather surprisingly, it turned out that it is not so; Rex and Rohn [18] proved that if (3.1) is valid, then A_c is nonsingular and

$$\varrho(|A_c^{-1}|\Delta) \leq \varrho(|I - RA_c| + |R|\Delta)$$

holds, hence (3.1) implies (3.2), so that both conditions cover the same class of interval matrices. This result also shows that the midpoint inverse A_c^{-1} is the best option for the choice of R. For related results, see Neumaier [11] and Rex [17].

Let us note that the condition (3.2) is verifiable in polynomial time since it is equivalent to

$$(I - |A_c^{-1}|\Delta)^{-1} \geq 0$$

and the inverse matrix can be evaluated in polynomial time by a modified Gaussian elimination (Bareiss [1]); this statement is of theoretical interest only since efficient numerical methods for checking (3.2) are available. The same reasoning applies also to (3.1) provided R is computed in polynomial time.

Next we prove a sufficient singularity condition of a similar type. Let A_j denote the jth column of a matrix A.

THEOREM 3.3. *Let there exist a matrix R such that*

$$(I + |I - A_cR|)_j \leq (\Delta|R|)_j \tag{3.3}$$

holds for some $j \in \{1, \ldots, n\}$. Then $[A_c - \Delta, A_c + \Delta]$ is singular.

Proof. The assumption (3.3) implies

$$\begin{aligned} |A_cR_j| &= |A_cR|_j = |I - (I - A_cR)|_j \leq I_j + |I - A_cR|_j \\ &\leq (\Delta|R|)_j = \Delta|R_j|, \end{aligned}$$

so that for $x := R_j$ we have

$$|A_cx| \leq \Delta|x|$$

where $x \neq 0$ due to (3.3), hence $[A_c - \Delta, A_c + \Delta]$ is singular by Theorem 2.1. ∎

Since the vector $x = R_j$ satisfies (2.3), we may employ the procedure described after Theorem 2.1 to construct a singular matrix contained in $[A_c - \Delta, A_c + \Delta]$. Setting $R = A_c^{-1}$, we immediately obtain as a special case a result from [20]:

COROLLARY 3.4. *Let A_c be nonsingular and let*

$$\max_j (\Delta |A_c^{-1}|)_{jj} \geq 1 \tag{3.4}$$

hold. Then $[A_c - \Delta, A_c + \Delta]$ is singular.

Proof. Let j be the index for which $(\Delta |A_c^{-1}|)_{jj} \geq 1$. Then (3.3) holds with $R = A_c^{-1}$, and Theorem 3.3 applies. ∎

During the time this paper was in reviewing process, Rump published a generalization of the condition (3.4): if

$$\max_{ij} (\Delta |A_c^{-1}|)_{ij} (\Delta |A_c^{-1}|)_{ji} \geq 1 \tag{3.5}$$

holds, then $[A_c - \Delta, A_c + \Delta]$ is singular [27, Thm. 6.5]. Obviously, (3.4) is a special case of (3.5) for $i = j$.

4. Sufficient conditions using eigenvalues. If A_c is nearly singular, then the conditions using approximate midpoint inverse may turn ineffective. Rump [26] was the first to derive a condition where no inverse matrix computation is required, at the expense of necessity to evaluate eigenvalues. Here we reprove his result by another means:

THEOREM 4.1. *Let*

$$\lambda_{\max}(\Delta^T \Delta) < \lambda_{\min}(A_c^T A_c) \tag{4.1}$$

hold. Then $[A_c - \Delta, A_c + \Delta]$ is regular.

Proof. Assume to the contrary that $[A_c - \Delta, A_c + \Delta]$ is singular, so that

$$|A_c x| \leq \Delta |x|$$

holds for some $x \neq 0$, which may be normalized to achieve $\|x\|_2 = 1$. Then we have

$$\begin{aligned} \lambda_{\min}(A_c^T A_c) &\leq x^T A_c^T A_c x \leq |A_c x|^T |A_c x| \leq (\Delta |x|)^T (\Delta |x|) \\ &= |x|^T \Delta^T \Delta |x| \leq \lambda_{\max}(\Delta^T \Delta), \end{aligned}$$

which contradicts (4.1). ∎

Let us note that the matrices $A_c^T A_c$ and $\Delta^T \Delta$ are symmetric, hence their eigenvalues appearing in (4.1) are real. Rump [29] and independently Vacek [30] found counterexamples demonstrating that neither of the conditions (3.2), (4.1) is a consequence of the other one.

The above result employed Theorem 2.1; using Theorem 2.2, we arrive at a sufficient singularity condition formulated in similar terms:

THEOREM 4.2. *Let*

$$\lambda_{\max}(A_c^T A_c) \leq \lambda_{\min}(\Delta^T \Delta) \tag{4.2}$$

hold. Then $[A_c - \Delta, A_c + \Delta]$ is singular.

Proof. Assume to the contrary that $[A_c - \Delta, A_c + \Delta]$ is regular. Then according to Theorem 2.2, applied to the nonnegative orthant $\mathcal{O}$, there exists an x satisfying

$$A_c x > \Delta |x|,$$

which can be normalized so that $\|x\|_2 = 1$. Then we have

$$\lambda_{\max}(A_c^T A_c) \geq x^T A_c^T A_c x > |x|^T \Delta^T \Delta |x| \geq \lambda_{\min}(\Delta^T \Delta)$$

contrary to (4.2). ∎

5. Sufficient conditions using positive definiteness. The necessity of evaluating eigenvalues in Theorems 4.1 and 4.2 may be avoided if we use instead a positive definiteness check. Let us recall that a symmetric matrix A (it will be seen that symmetry poses no restriction here) is positive definite if and only if all its leading principal minors are positive (Sylvester determinant criterion, see Wilkinson [31]). Since positivity of all leading principal minors may be checked by employing a modified Gaussian elimination which is performable in polynomial time (Bareiss [1]), we can see that checking positive definiteness of symmetric matrices may be performed by a polynomial-time algorithm. This is the advantage of criteria presented in this section; their disadvantage consists in the fact that they require evaluation of $A_c^T A_c$ (or $\Delta^T \Delta$), which squares the condition number.

THEOREM 5.1. *Let the matrix*

$$A_c^T A_c - \|\Delta^T \Delta\| I \tag{5.1}$$

be positive definite for some consistent matrix norm $\|\cdot\|$. *Then* $[A_c - \Delta, A_c + \Delta]$ *is regular.*

Proof. As in the proof of Theorem 4.1, assuming to the contrary that $[A_c - \Delta, A_c + \Delta]$ is singular, we may assure existence of an x with $\|x\|_2 = 1$ satisfying

$$x^T A_c^T A_c x \le |x|^T \Delta^T \Delta |x| \le \lambda_{\max}(\Delta^T \Delta) = \varrho(\Delta^T \Delta) \le \|\Delta^T \Delta\| = \|\Delta^T \Delta\|(x^T x),$$

hence

$$x^T (A_c^T A_c - \|\Delta^T \Delta\| I) x \le 0,$$

which means that the matrix (5.1) is not positive definite, a contradiction. □

Notice that the matrix (5.1) is symmetric, which justifies the discussion made at the beginning of this section. Since $\|\Delta^T \Delta\| \le \|\Delta^T\| \cdot \|\Delta\|$, Theorem 5.1 will remain valid if we replace (5.1) by the matrix

$$A_c^T A_c - \|\Delta^T\| \cdot \|\Delta\| I,$$

which yields a weaker result where, however, $\Delta^T \Delta$ need not be computed. We note that any of the usual matrix norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$ and $\|\cdot\|_F$ is consistent [5] and may be employed in Theorem 5.1. The theorem will not stay in force if the matrix (5.1) is replaced by

$$A_c^T A_c - \Delta^T \Delta.$$

Indeed, for

$$A_c = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}, \qquad \Delta = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

the matrix $A_c^T A_c - \Delta^T \Delta = 9I$ is positive definite, but $[A_c - \Delta, A_c + \Delta]$ contains the singular matrix

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

(Rump [29]). Finally, we formulate in similar terms a sufficient singularity condition:

THEOREM 5.2. *Let the matrix*

$$\Delta^T\Delta - A_c^T A_c \tag{5.2}$$

be positive semidefinite. Then $[A_c - \Delta, A_c + \Delta]$ *is singular.*

Proof. Assume to the contrary that $[A_c - \Delta, A_c + \Delta]$ is regular. Then Theorem 2.2 (applied to the nonnegative orthant) implies existence of an $x \neq 0$ satisfying

$$A_c x > \Delta|x|$$

and henceforth also

$$x^T A_c^T A_c x > |x|^T \Delta^T \Delta |x| \geq x^T \Delta^T \Delta x,$$

which means that

$$x^T(\Delta^T\Delta - A_c^T A_c)x < 0$$

and the matrix (5.2) is not positive semidefinite, which contradicts the assumption. ∎

6. Application: Condition for an interval matrix to have real eigenvalues only. As an application of the above results, different from those mentioned in the introduction, we shall consider the problem of checking that each $A \in A^I$ has real eigenvalues only. The single reference on this problem known to us is the paper by Hollot and Bartlett [6]; the necessary and sufficient condition given there, however, is not of practical use since it is exponential in the matrix size. We have this verifiable sufficient condition:

THEOREM 6.1. *Let* A_c *have* n *simple real eigenvalues*

$$\lambda_1(A_c) < \lambda_2(A_c) < \ldots < \lambda_n(A_c)$$

and let there exist real numbers $\mu_0, \ldots, \mu_n$ *satisfying*

$$\mu_0 < \lambda_1(A_c) < \mu_1 < \lambda_2(A_c) < \mu_2 < \ldots < \lambda_n(A_c) < \mu_n \tag{6.1}$$

such that the interval matrix

$$[A_c - \mu_j I - \Delta, A_c - \mu_j I + \Delta] \tag{6.2}$$

is regular for $j = 0, \ldots, n$. *Then each* $A \in A^I$ *has* n *simple real eigenvalues satisfying*

$$\mu_0 < \lambda_1(A) < \mu_1 < \lambda_2(A) < \mu_2 < \ldots < \lambda_n(A) < \mu_n. \tag{6.3}$$

Proof. For an $A \in A^I$, let

$$p(\lambda) = \det(A - \lambda I)$$

denote its characteristic polynomial and let

$$p_c(\lambda) = \det(A_c - \lambda I)$$

be the characteristic polynomial of A_c. Then for each $j \in \{0, \ldots, n\}$ we have $|(A - \mu_j I) - (A_c - \mu_j I)| = |A - A_c| \leq \Delta$, hence

$$A - \mu_j I \in [A_c - \mu_j I - \Delta, A_c - \mu_j I + \Delta],$$

and regularity of (6.2) implies

$$p(\mu_j)p_c(\mu_j) > 0 \tag{6.4}$$

since $p(\mu_j)p_c(\mu_j) \leq 0$ would imply, by continuity of the determinant, existence of a singular matrix in (6.2), a contradiction. Now, since all eigenvalues of A_c are real and simple, (6.1) gives

$$p_c(\mu_j)p_c(\mu_{j+1}) < 0 \tag{6.5}$$

for $j = 0, \dots, n-1$. For each such j we have from (6.4)

$$p(\mu_j)p_c(\mu_j)p(\mu_{j+1})p_c(\mu_{j+1}) > 0,$$

which in view of (6.5) implies

$$p(\mu_j)p(\mu_{j+1}) < 0,$$

hence the characteristic polynomial of A has a root in each of the open intervals (μ_j, μ_{j+1}), $j = 0, \dots, n-1$. This proves that A has exactly n simple real eigenvalues satisfying (6.3). $\square$

Regularity of the interval matrix (6.2) may be checked by any of the sufficient regularity conditions presented above. Theorem 5.1 seems to be particularly suited here since it requires checking positive definiteness of the matrices

$$(A_c - \mu_j I)^T(A_c - \mu_j I) - \|\Delta^T\Delta\|I = A_c^T A_c - \mu_j(A_c^T + A_c) + (\mu_j^2 - \|\Delta^T\Delta\|)I$$

that may be easily updated for different values of μ_j.

7. Concluding remarks. In a very recent development, Jansson [7] proposed a necessary and sufficient regularity condition, based on a quite different idea, which is not *a priori* exponential (an exponential growth occurs only at worst-case-type examples). Computational results reported in [8] look promising: interval matrices up to the size $n = 50$ were checked in acceptable time. Nevertheless, in view of the NP-hardness result and of the famous conjecture "P≠NP" (see Garey and Johnson [3] for details) there remains only very little hope that necessary and sufficient regularity conditions verifiable in polynomial time might be found.

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