Complexity of Some Linear Problems with Interval Data

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Abstract. During the recent years, a number of linear problems with interval data have been proved to be NP–hard. These results may seem rather obscure as regards the ways in which they were obtained. This survey paper is aimed at demonstrating that in fact it is not so, since many of these results follow easily from the recently established fact that for the subordinate matrix norm $\|\cdot\|_{\infty,1}$ it is NP–hard to decide whether $\|A\|_{\infty,1} \geq 1$ holds, even in the class of symmetric positive definite rational matrices. After a brief introduction into the basic topics of the complexity theory in section 1 and formulation of the underlying norm complexity result in section 2, we present NP–hardness results for checking properties of interval matrices (section 3), computing enclosures (section 4), solvability of rectangular linear interval systems (section 5), and linear and quadratic programming (section 6). Due to space limitations, proofs are mostly only sketched to reveal the unifying role of the norm complexity result; technical details are omitted.

Key words: Interval Data, Norm, Checking, Enclosure, Solvability, Linear Programming, Quadratic Programming

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1. Complexity

An algorithm is called a polynomial–time algorithm if there exists a polynomial p such that for each instance (input data) of length ℓ the number of steps of the algorithm is $\leq p(\ell)$. Length: number of bits of the input. Consequence: only rational data allowed (usually represented by pairs of integers). Example: modified Gaussian elimination [1].

Decision ("yes or no") problems are considered in complexity theory. A problem belongs to the class P if it is solvable by a polynomial–time algorithm, and to the class NP if a guessed candidate for a solution can be verified by a polynomial–time algorithm.

A problem I can be reduced in polynomial time to problem J, which we denote by $I \to J$, if there exists a polynomial–time algorithm π which transforms each instance i of I to an instance $\pi(i)$ of J so that the answer to i is "yes" if and only if the answer to $\pi(i)$ is "yes" (or, the answer to i is "yes" if and only if the answer to $\pi(i)$ is "no"). Hence, if $I \to J$, then each algorithm for solving J may be employed for solving I; consequently, J is "at least as difficult" as I.

A problem J is called NP–hard if $I \to J$ for each $I \in NP$. An NP–hard problem exists (Cook [2]; hundreds of them have been found since). Method for proving NP–hardness: if J is NP–hard and $J \to K$, then K is NP–hard.

Computing the value of

$$\max_{x \in X} f(x)$$

is said to be NP–hard if the decision problem

"is $f(x) \geq r$ for some $x \in X$?"

is NP–hard (r rational).

If *some* NP–hard problem can be solved by a polynomial–time algorithm, then *all* problems in NP are solvable by polynomial–time algorithms. This would imply $P = NP$. However, *no* such problem (or algorithm) is known to date, and it is widely believed (but not proved) that

$$P \neq NP.$$

Hence, if this conjecture is true, then no NP–hard problem can be solved by a polynomial–time algorithm. For more details, see Garey and Johnson [3].

2. The norm $\|A\|_{\infty,1}$

Given two vector norms $\|x\|_\alpha$ and $\|x\|_\beta$ in R^n, a subordinate matrix norm in $R^{n \times n}$ is defined by

$$\|A\|_{\alpha,\beta} = \max_{\|x\|_\alpha = 1} \|Ax\|_\beta$$

(see Golub and van Loan [6]). We shall consider the particular norm

$$\|A\|_{\infty,1} = \max_{\|x\|_\infty = 1} \|Ax\|_1$$

(where $\|x\|_\infty = \max_i |x_i|$ and $\|x\|_1 = \sum_i |x_i|$).

THEOREM 1. [17] *For each $A \in R^{n \times n}$ we have*

$$\|A\|_{\infty,1} = \max\{\|Ax\|_1;\ |x| = e\},$$

where $e = (1, 1, \ldots, 1)^T$. Moreover, if A is symmetric positive definite, then

$$\|A\|_{\infty,1} = \max\{x^T Ax;\ |x| = e\}.$$

Note 1. Both finite formulae require maximization over the set of all ± 1-vectors (of cardinality 2^n).

THEOREM 2. [12], [14] *The following decision problem is NP–hard:*
Instance. *A symmetric positive definite rational matrix A.*
Question. *Is $\|A\|_{\infty,1} \geq 1$?*

Proof sketch. The NP–hard problem "simple max–cut in a graph" [4] can be reduced in polynomial time to this one; see [17] for a detailed proof. □

COROLLARY 1. [14] *The problem of checking*

$$\|A\|_{\infty,1} < 1$$

is also NP–hard.

COROLLARY 2. [14] *Computing $\|A\|_{\infty,1}$ is NP–hard even in the class of symmetric positive definite rational matrices.*

3. Checking properties

A square interval matrix

$$A^I = [\underline{A}, \overline{A}] = \{A;\ \underline{A} \leq A \leq \overline{A}\}$$

is said to be

- *regular* if each $A \in A^I$ is nonsingular,

and

- *positive definite,*
- *P-matrix,*
- *stable*

if each $A \in A^I$ has the respective property.

PROPOSITION 1. [17] *For a symmetric positive definite matrix A, let*

$$A^I := [A^{-1} - E, A^{-1} + E],$$

where E is the matrix of all ones. Then the following assertions are equivalent:

(i) $\|A\|_{\infty,1} < 1$,

(ii) A^I *is regular,*

(iii) A^I *is positive definite,*

(iv) A^I *is a* P*-matrix,*

(v) $[-A^{-1} - E, -A^{-1} + E]$ *is stable.*

Proof sketch. (ii)⇒(i) by contradiction: if

$$\|A\|_{\infty,1} = x^T Ax \geq 1$$

for some x with $|x| = e$, then

$$A' := A^{-1} - \frac{xx^T}{x^T Ax}$$

belongs to A^I and is singular (since $A'Ax = 0$).

(i)⇒(ii) by contradiction: if

$$A''y = 0$$

for some $A'' \in A^I$ and $y \neq 0$, then for the sign vector x of y we have

$$1 \leq \|Ax\|_1 \leq \|A\|_{\infty,1}.$$

(ii)⇔(n)∈{(iii),(iv),(v)} by elementary means. □

THEOREM 3. [12], [13], [20] *Checking*

- *regularity,*
- *positive definiteness,*
- P*-property,*
- *stability*

is NP–hard even in the class of rational interval matrices of the form

$$[B - E, B + E], \tag{1}$$

where B *is symmetric positive definite and* E *is the matrix of all ones.*

Proof. According to the previous proposition, the NP–hard problem

$$''\|A\|_{\infty,1} < 1''$$

can be reduced in polynomial time [1] to checking any of these properties for interval matrices of the form (1). □

4. Computing enclosures

For a system of linear interval equations

$$A^I x = b^I$$

(A^I square), *optimal enclosure* is defined as the narrowest interval vector $[\underline{x}, \overline{x}]$ satisfying

$$X \subseteq [\underline{x}, \overline{x}],$$

where

$$X = \{x;\ Ax = b \text{ for some } A \in A^I,\ b \in b^I\}$$

is the solution set. If A^I is regular, then the optimal enclosure is given by

$$\begin{aligned} \underline{x}_i &= \min_X x_i, \\ \overline{x}_i &= \max_X x_i \end{aligned}$$

$(i = 1, \ldots, n)$.

PROPOSITION 2. [16] *Let A be symmetric positive definite and let $\varepsilon > 0$. Then for the linear interval system*

$$A^I x = b^I \tag{2}$$

given by

$$A^I = \begin{pmatrix} 1 & [-\varepsilon e^T, \varepsilon e^T] \\ 0 & A^{-1} \end{pmatrix}, \tag{3}$$

$$b^I = \begin{pmatrix} 0 \\ [-\varepsilon e, \varepsilon e] \end{pmatrix} \tag{4}$$

(e is the vector of all ones) we have

$$\overline{x}_1 = \varepsilon^2 \|A\|_{\infty,1}.$$

Moreover, $A^I = [A_c - \Delta, A_c + \Delta]$ satisfies

$$\varrho(|A_c^{-1}|\Delta) = 0$$

(i.e., is strongly regular).

Proof sketch.

$$\begin{aligned} \overline{x}_1 &= \max\{\varepsilon e^T |x|;\ -\varepsilon e \le A^{-1}x \le \varepsilon e\} = \varepsilon^2 \max\{\|Ax'\|_1;\ -e \le x' \le e\} \\ &= \varepsilon^2 \max\{\|Ax'\|_1;\ \|x'\|_\infty = 1\} = \varepsilon^2 \|A\|_{\infty,1}. \end{aligned}$$

□

THEOREM 4. [18] *For each $\varepsilon > 0$, computing the optimal enclosure $[\underline{x}, \overline{x}]$ is NP–hard even in the class of rational systems of the form (2)–(4).*

Proof. According to the previous proposition,

$$\|A\|_{\infty,1} = \frac{\overline{x}_1}{\varepsilon^2},$$

hence the NP–hard problem of computing $\|A\|_{\infty,1}$ is reduced in polynomial time [1] to that of computing $\overline{x}_1$. □

5. Solvability (rectangular case)

Under a system of linear interval equations

$$A^I x = b^I \tag{5}$$

(A^I of size $m \times n$) we understand the family of systems

$$Ax = b \tag{6}$$

with data satisfying

$$A \in A^I,\ b \in b^I. \tag{7}$$

A system (5) is called

- *strongly feasible* if each system (6) with data (7) has a nonnegative solution,
- *weakly feasible* if some system (6) with data (7) has a nonnegative solution.

THEOREM 5. [15] *We have:*

(i) checking strong feasibility is NP–hard even in the case $n = 2m$,

(ii) checking weak feasibility can be performed in polynomial time since it is equivalent to solvability of

$$\underline{A}x \le \overline{b},\ \overline{A}x \ge \underline{b},\ x \ge 0.$$

Proof sketch. (i) For a symmetric matrix A,

$$\|A\|_{\infty,1} < 1$$

is equivalent to strong solvability of the system (5) with

$$\begin{aligned} A^I &= [(-A - E, A - E),\ (-A + E, A + E)], \\ b^I &= [-e, e] \end{aligned}$$

(by Farkas lemma).

(ii) Due to Oettli–Prager theorem [11], (5) is weakly solvable if and only if

$$\underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0$$

has a solution, which can be checked by a polynomial-time linear programming algorithm (Khachiyan [7]). □

The problem of checking weak feasibility of (5) *without* the nonnegativity restriction was proved to be NP–hard by Lakeyev and Kreinovich [9].

As before, under a system of linear interval inequalities

$$A^I x \leq b^I \tag{8}$$

(A^I of size $m \times n$) we understand the family of systems

$$Ax \leq b \tag{9}$$

with data satisfying

$$A \in A^I,\ b \in b^I. \tag{10}$$

A system (8) is called

- *strongly solvable* if each system (9) with data (10) has a solution,
- *weakly solvable* if some system (9) with data (10) has a solution.

THEOREM 6. [19], [14] *We have:*

(i) checking strong solvability can be performed in polynomial time since it is equivalent to solvability of

$$\overline{A}x_1 - \underline{A}x_2 \leq \underline{b},$$

$$x_1 \geq 0,\ x_2 \geq 0,$$

(ii) checking weak solvability is NP–hard even in the case $m = 2n + 1$.

Proof sketch. (i) The equivalence was proved by Rohn and Kreslová [19].

(ii) Due to Gerlach's description of weak solutions [5], for a symmetric positive definite A,

$$\|A\|_{\infty,1} \geq 1$$

is equivalent to weak solvability of

$$A^I x \leq b^I,$$

where

$$A^I = \begin{pmatrix} A^{-1} \\ -A^{-1} \\ [-e^T, e^T] \end{pmatrix},$$

$$b^I = \begin{pmatrix} e \\ e \\ -1 \end{pmatrix}.$$

□

6. Linear and quadratic programming

For a linear programming problem

$$\text{minimize } c^T x$$

subject to

$$Ax = b,\ x \geq 0$$

denote

$$f(A, b, c) = \inf\{c^T x;\ Ax = b, x \geq 0\}.$$

Consider the problem with fixed A, c and inexact right-hand side $b \in [\underline{b}, \overline{b}]$. Let

$$\begin{aligned} \underline{f} &= \inf\{f(A, b, c);\ b \in [\underline{b}, \overline{b}]\}, \\ \overline{f} &= \sup\{f(A, b, c);\ b \in [\underline{b}, \overline{b}]\} \end{aligned}$$

(range of optimal value; the bounds may be infinite).

THEOREM 7. [10], [14] *We have:*

(i) computing $\underline{f}$ can be done in polynomial time since

$$\underline{f} = \inf\{c^T x;\ \underline{b} \leq Ax \leq \overline{b}, x \geq 0\},$$

(ii) computing $\overline{f}$ is NP–hard even in the case of $n = 2m$ and of a finite value of $\overline{f}$.

Proof sketch. (i) is known (Mráz [10]).

(ii) Given a symmetric positive definite A, for the problem

$$\text{minimize } e^T x_1 + e^T x_2$$

subject to

$$A^{-1}x_1 - A^{-1}x_2 = b,\ x_1 \geq 0,\ x_2 \geq 0$$

with

$$-e \leq b \leq e$$

we have

$$\overline{f} = \|A\|_{\infty,1}$$

(by duality theorem). □

THEOREM 8. [8], [14] *Let D be symmetric positive definite. Then*

(i) the problem of computing

$$\min\{x^T Dx + c^T x;\ Ax \leq b\}$$

can be solved in polynomial time,

(ii) the problem of computing

$$\max\{x^T Dx + c^T x;\ Ax \leq b\}$$

is NP–hard even in the case $m = 2n$.

Proof sketch. (i) is known (Kozlov, Tarasov and Khachiyan [8]).
(ii) We have

$$\begin{aligned} \|D\|_{\infty,1} &= \max\{x^T Dx;\ |x| = e\} \\ &= \max\left\{x^T Dx;\ \begin{pmatrix} I \\ -I \end{pmatrix} x \leq \begin{pmatrix} e \\ e \end{pmatrix}\right\}. \end{aligned}$$

□

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