Rohn, J.

Bounds on Eigenvalues of Interval Matrices

We describe a rectangle in the complex plane enclosing all eigenvalues of an interval matrix. We give theoretical bounds (Theorem 1) that are exact for symmetric or skew-symmetric matrices, and practical bounds (Theorem 2) requiring evaluation of 4 minimal or maximal eigenvalues and 2 spectral radii of symmetric matrices.

1. Theoretical bounds

We consider square interval matrices in the form $A^I = [A_c - \Delta, A_c + \Delta] = \{A;\ A_c - \Delta \le A \le A_c + \Delta\}$ where inequalities are understood componentwise; thus A_c is the center matrix and Δ is the radius matrix of A^I.

Theorem 1. *Let $A^I = [A_c - \Delta, A_c + \Delta]$ be a square interval matrix. Then for each eigenvalue λ of each $A \in A^I$ we have*

$$\underline{r} \le \operatorname{Re}\lambda \le \overline{r}, \tag{1}$$

$$\underline{\imath} \le \operatorname{Im}\lambda \le \overline{\imath}, \tag{2}$$

where

$$\begin{aligned}
\underline{r} &= \min_{\|x\|_2=1} (x^T A_c x - |x|^T \Delta |x|),\\
\overline{r} &= \max_{\|x\|_2=1} (x^T A_c x + |x|^T \Delta |x|),\\
\underline{\imath} &= \min_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T|),\\
\overline{\imath} &= \max_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|).
\end{aligned}$$

Remark. Vectors are always considered column vectors, so that $x^T y$ is the scalar product whereas xy^T is the matrix $(x_i y_j)$. In the formulae for $\underline{\imath}$ and $\overline{\imath}$, for typographic reasons we write "$\|(x_1, x_2)\|_2 = 1$" in the subscript instead of the correct "$\|(x_1^T, x_2^T)^T\|_2 = 1$". For $A = (a_{ij})$ and $B = (b_{ij})$ we use $A \circ B = \sum_{ij} a_{ij} b_{ij}$ ("scalar product of matrices") and $|A| = (|a_{ij}|)$. Then we have $x^T A y = \sum_{ij} a_{ij} x_i y_j = A \circ (xy^T)$.

Proof. Let $\lambda = \lambda_1 + \lambda_2 i$ be an eigenvalue of some $A \in A^I$. Then $A(x_1 + x_2 i) = (\lambda_1 + \lambda_2 i)(x_1 + x_2 i)$ for some real vectors x_1, x_2, $x_1 \ne 0$ or $x_2 \ne 0$, that may be normalized to achieve $x_1^T x_1 + x_2^T x_2 = 1$. Premultiplying by the complex conjugate vector $x_1 - x_2 i$, we obtain $\lambda_1 + \lambda_2 i = (x_1 - x_2 i)^T A (x_1 + x_2 i)$, which yields $\operatorname{Re}\lambda = \lambda_1 = x_1^T A x_1 + x_2^T A x_2$ and $\operatorname{Im}\lambda = \lambda_2 = x_1^T A x_2 - x_2^T A x_1$. 1) To prove that $\operatorname{Re}\lambda \le \overline{r}$, denote $r(A) = \max_{\|x\|_2=1} x^T A x$, then we have $x_1^T A x_1 \le r(A) x_1^T x_1$ and $x_2^T A x_2 \le r(A) x_2^T x_2$, hence $x_1^T A x_1 + x_2^T A x_2 \le r(A)(x_1^T x_1 + x_2^T x_2) = r(A) = \max_{\|x\|_2=1} x^T A x = \max_{\|x\|_2=1} (x^T A_c x + x^T (A - A_c) x) \le \max_{\|x\|_2=1} (x^T A_c x + |x|^T \Delta |x|) = \overline{r}$. Then $\operatorname{Re}\lambda \le \overline{r}$, which is the right-hand side inequality in (1). 2) Since $-\lambda$ is an eigenvalue of $-A$ which belongs to $[-A_c - \Delta, -A_c + \Delta]$, from the result proved in 1) applied to $[-A_c - \Delta, -A_c + \Delta]$ we obtain $-\operatorname{Re}\lambda = \operatorname{Re}(-\lambda) \le \max_{\|x\|_2=1} (-x^T A_c x + |x|^T \Delta |x|)$, which implies $\operatorname{Re}\lambda \ge -\max_{\|x\|_2=1} (-x^T A_c x + |x|^T \Delta |x|) = \min_{\|x\|_2=1} (x^T A_c x - |x|^T \Delta |x|) = \underline{r}$, which is the left-hand side inequality in (1). 3) From $x_1^T A x_2 - x_2^T A x_1 = x_1^T (A_c - A_c^T) x_2 + (A - A_c) \circ (x_1 x_2^T - x_2 x_1^T) \le x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|$ we get $\operatorname{Im}\lambda \le \max_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|) = \overline{\imath}$, which is the right-hand side inequality in (2). 4) Since $-\lambda$ is an eigenvalue of $-A \in [-A_c - \Delta, -A_c + \Delta]$, applying the result in 3) we obtain $-\operatorname{Im}\lambda = \operatorname{Im}(-\lambda) \le \max_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c^T - A_c) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|)$ and thereby also $\operatorname{Im}\lambda \ge \min_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T|) = \underline{\imath}$, which concludes the proof. ∎

An interval matrix A^I is called symmetric if both A_c and Δ are symmetric, and it is called skew-symmetric if A_c is skew-symmetric and Δ is symmetric. *The bounds (1) are exact (i.e., achieved over A^I) if A^I is symmetric and the bounds (2) are exact if A^I is skew-symmetric.* The proof of this assertion is omitted here due to space limitations.

2. Practical bounds

Theorem 2. *Let $A^I = [A_c - \Delta, A_c + \Delta]$ be a square interval matrix. Then for each eigenvalue λ of each $A \in A^I$ we have*

$$\lambda_{\min}(A'_c) - \varrho(\Delta') \le \operatorname{Re}\lambda \le \lambda_{\max}(A'_c) + \varrho(\Delta'), \tag{3}$$

$$\lambda_{\min}(A''_c) - \varrho(\Delta'') \le \operatorname{Im}\lambda \le \lambda_{\max}(A''_c) + \varrho(\Delta''), \tag{4}$$

where

$$\begin{aligned} A'_c &= \tfrac{1}{2}(A_c + A_c^T), \\ \Delta' &= \tfrac{1}{2}(\Delta + \Delta^T), \\ A''_c &= \begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix}, \\ \Delta'' &= \begin{pmatrix} 0 & \Delta' \\ \Delta' & 0 \end{pmatrix}. \end{aligned}$$

Remark. $\lambda_{\min}, \lambda_{\max}$ denote the minimal and maximal eigenvalue of a symmetric matrix, respectively, and ϱ is the spectral radius. Notice that all the matrices A'_c, Δ', A''_c, Δ'' are symmetric by definition.

Proof. Let λ be an eigenvalue of a matrix $A \in A^I$. 1) Since $\overline{r} = \max_{\|x\|_2=1}(x^T A_c x + |x|^T \Delta |x|) \le \max_{\|x\|_2=1} x^T A_c x + \max_{\|x\|_2=1} |x|^T \Delta |x| = \max_{\|x\|_2=1} x^T A'_c x + \max_{\|x\|_2=1} |x|^T \Delta' |x| = \lambda_{\max}(A'_c) + \lambda_{\max}(\Delta') = \lambda_{\max}(A'_c) + \varrho(\Delta')$, by Theorem 1 there holds $\operatorname{Re}\lambda \le \lambda_{\max}(A'_c) + \varrho(\Delta')$, which is the right-hand side inequality in (3). 2) The proof of the left-hand side inequality is analogous since $\underline{r} \ge \min_{\|x\|_2=1} x^T A_c x - \max_{\|x\|_2=1} |x|^T \Delta |x| = \lambda_{\min}(A'_c) - \varrho(\Delta')$. 3) We have

$$\begin{aligned} \overline{\imath} &= \max_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|) \\ &\le \max_{\|(x_1,x_2)\|_2=1} (x_1^T A_c x_2 - x_2^T A_c x_1) + \max_{\|(x_1,x_2)\|_2=1} (|x_1|^T \Delta |x_2| + |x_2|^T \Delta |x_1|) \\ &= \max_{\|(x_1,x_2)\|_2=1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\quad + \max_{\|(x_1,x_2)\|_2=1} \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix}^T \begin{pmatrix} 0 & \frac{1}{2}(\Delta + \Delta^T) \\ \frac{1}{2}(\Delta + \Delta^T) & 0 \end{pmatrix} \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix} \\ &= \lambda_{\max}(A''_c) + \lambda_{\max}(\Delta'') = \lambda_{\max}(A''_c) + \varrho(\Delta''). \end{aligned}$$

Hence Theorem 1 implies $\operatorname{Im}\lambda \le \overline{\imath} \le \lambda_{\max}(A''_c) + \varrho(\Delta'')$, which is the right-hand side inequality in (4). 4) An analogous reasoning gives

$$\begin{aligned} \underline{\imath} &\ge \min_{\|(x_1,x_2)\|_2=1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\quad - \max_{\|(x_1,x_2)\|_2=1} \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix}^T \begin{pmatrix} 0 & \frac{1}{2}(\Delta + \Delta^T) \\ \frac{1}{2}(\Delta + \Delta^T) & 0 \end{pmatrix} \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix} \\ &= \lambda_{\min}(A''_c) - \varrho(\Delta''), \end{aligned}$$

which in view of Theorem 1 implies the left-hand side inequality in (4). ■

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3. References

1 Rohn, J.: Preliminary announcement (circular letter), March 1996.

Addresses: Jiří Rohn, Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 118 00 Prague, and Institute of Computer Science, Academy of Sciences, Pod vodárenskou věží 2, 182 07 Prague, Czech Republic.