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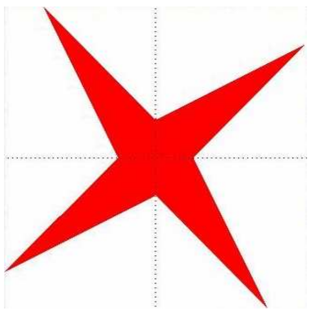
Abstract:

We describe an iterative method for solving absolute value equations. The usual spectral condition is replaced by assumption of existence of two matrices satisfying certain matrix inequality.



Keywords:
Absolute value equation, iterative method.[5]

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[5]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

1 Introduction

The absolute value equation

$$Ax + B|x| = b \tag{1.1}$$

(where $A, B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$) has been recently studied by several authors, cf. e.g. Caccetta, Qu and Zhou [2], Hu and Huang [3], Karademir and Prokopyev [4], Mangasarian [5], [6], Mangasarian and Meyer [7], Prokopyev [8], Rohn [9], [10], and Zhang and Wei [11]. Little attention was, however, dedicated so far to iterative methods for solving (1.1). In this report we formulate such a method (Theorem 1) working under assumption of existence of matrices $M \geq 0$ and R satisfying the inequality

$$M(I - |I - RA| - |RB|) \geq I \tag{1.2}$$

which replaces the traditional spectral condition. This inequality is discussed in more detail in Sections 4 and 5, and interval iterations are proposed in Section 3.

2 The method

The following theorem is the basic result of this report.

Theorem 1. *Let $M \geq 0$ and R satisfy (1.2). Then the sequence $\{x^i\}_{i=0}^{\infty}$ given by $x^0 = Rb$ and*

$$x^{i+1} = (I - RA)x^i - RB|x^i| + Rb \qquad (i = 0, 1, \ldots) \tag{2.1}$$

converges to the unique solution x^* of (1.1) and for each $i \geq 1$ there holds

$$|x^* - x^i| \leq (M - I)|x^i - x^{i-1}|. \tag{2.2}$$

Proof. Let (1.2) have a solution $M \geq 0$ and R. Denote

$$G = |I - RA| + |RB|,$$

then $G \geq 0$ and the condition (1.2) can be written as

$$I + MG \leq M. \tag{2.3}$$

Postmultiplying this inequality by G and adding I to both sides we obtain

$$I + G + MG^2 \leq I + MG \leq M$$

and by induction

$$\sum_{j=0}^{k} G^j + MG^{k+1} \leq M$$

for $k = 0, 1, 2, \ldots$. In view of nonnegativity of M, this shows that the nonnegative matrix series $\sum_{j=0}^{\infty} G^j$ satisfies

$$\sum_{j=0}^{\infty} G^j \leq M, \tag{2.4}$$

hence it is convergent, so that $G^j \to 0$ and consequently

$$\varrho(G) < 1. \tag{2.5}$$

Now we have

$$I - RA \leq |I - RA| \leq G,$$

hence

$$\varrho(I - RA) \leq \varrho(|I - RA|) \leq \varrho(G) < 1. \tag{2.6}$$

Since $\varrho(I - RA) < 1$, the matrix

$$RA = I - (I - RA) \tag{2.7}$$

is nonsingular, which gives that both R and A are nonsingular.

Let $i \geq 1$. Subtracting the equations

$$\begin{aligned} x^{i+1} &= (I - RA)x^i - RB|x^i| + Rb, \\ x^i &= (I - RA)x^{i-1} - RB|x^{i-1}| + Rb, \end{aligned}$$

we get

$$|x^{i+1} - x^i| \leq |I - RA||x^i - x^{i-1}| + |RB|||x^i| - |x^{i-1}|| \leq G|x^i - x^{i-1}|$$

and for each $m \geq 1$ by induction

$$\begin{aligned} |x^{i+m} - x^i| &= |\sum_{j=0}^{m-1}(x^{i+j+1} - x^{i+j})| \leq \sum_{j=0}^{m-1}|x^{i+j+1} - x^{i+j}| \leq \sum_{j=0}^{m-1} G^{j+1}|x^i - x^{i-1}| \\ &\leq (\sum_{j=0}^{\infty} G^{j+1})|x^i - x^{i-1}| \leq (M - I)|x^i - x^{i-1}| \leq (M - I)G^{i-1}|x^1 - x^0|. \end{aligned}$$

From the final inequality

$$|x^{i+m} - x^i| \leq (M - I)G^{i-1}|x^1 - x^0|,$$

in view of the fact that $G^{i-1} \to 0$ as $i \to \infty$, we can see that the sequence $\{x^i\}$ is Cauchian, thus convergent, $x^i \to x^*$. Taking the limit in (2.1) we get

$$x^* = (I - RA)x^* - RB|x^*| + Rb$$

which gives

$$0 = R(Ax^* + B|x^*| - b),$$

and employing the above-proved nonsingularity of R, we can see that x^* is a solution of (1.1). The estimation (2.2) follows from the above-established inequality

$$|x^{i+m} - x^i| \leq (M - I)|x^i - x^{i-1}|$$

by taking $m \to \infty$.

To prove uniqueness, assume that (1.1) has a solution x'. From the equations

$$\begin{aligned} Ax^* + B|x^*| &= b, \\ Ax' + B|x'| &= b \end{aligned}$$

written as

$$\begin{aligned} x^* &= (I - RA)x^* - RB|x^*| + Rb, \\ x' &= (I - RA)x' - RB|x'| + Rb \end{aligned}$$

we obtain

$$|x^* - x'| \le G|x^* - x'|,$$

hence

$$(I - G)|x^* - x'| \le 0$$

and by (2.3),

$$|x^* - x'| \le M(I - G)|x^* - x'| \le 0,$$

which shows that $x' = x^*$. This proves uniqueness. ∎

Employing the approach used in the last part of the proof, we can derive an estimation of the distance of the solution from an arbitrary point in $\mathbb{R}^n$.

Theorem 2. *Let $M \ge 0$ and R satisfy (1.2). Then for the unique solution x^* of (1.1) there holds*

$$|x^* - x| \le M|R(Ax + B|x| - b)| \tag{2.8}$$

for each $x \in \mathbb{R}^n$.

Proof. Let (1.2) have a solution $M \ge 0$ and R. Denote

$$G = |I - RA| + |RB|.$$

Then $G \ge 0$ and the condition (1.2) can be written as

$$I \le M(I - G). \tag{2.9}$$

Now take an arbitrary $x \in \mathbb{R}^n$ and put

$$Ax + B|x| - b = r,$$

then premultiplying this equality by R and adding x to both sides we obtain

$$x = (I - RA)x - RB|x| + Rb + Rr.$$

Since x^* is the unique solution of (1.1) we have

$$Ax^* + B|x^*| = b,$$

which similarly implies that

$$x^* = (I - RA)x^* - RB|x^*| + Rb.$$

Hence

$$|x^* - x| \le |I - RA||x^* - x| + |RB||x^* - x| + |Rr| = G|x^* - x| + |Rr|.$$

Using relation 2.9 we have

$$|x^* - x| \le M(I - G)|x^* - x| \le M|Rr|.$$

Consequently

$$|x^* - x| \le M|R(Ax + B|x| - b)|$$

which was to be proved. ∎

The result can also be formulated in an explicit interval form.

Theorem 3. *Under assumptions and notation of Theorem 2 we have*

$$x^* \in [x - M|R(Ax + B|x| - b)|,\, x + M|R(Ax + B|x| - b)|] \tag{2.10}$$

for each $x \in \mathbb{R}^n$.

Proof. Clearly, the relation (2.8) can be equivalently rewritten as

$$x - M|R(Ax + B|x| - b)| \le x^* \le x + M|R(Ax + B|x| - b)|,$$

which is (2.10). ∎

3 Interval iterations

When generating the sequence (2.1) on computer, instead of the true sequence

$$x^0, x^1, \dots, x^i, \dots$$

we compute a floating-point sequence

$$\tilde{x}^0, \tilde{x}^1, \dots, \tilde{x}^i, \dots$$

for which the estimation (2.2) no longer remains in force. To overcome this difficulty, we may resort to interval iterations. For each $i \ge 0$, using outward rounding (e.g. in INTLAB), first compute verified enclosures

$$\tilde{x}^i - M|R(A\tilde{x}^i + B|\tilde{x}^i| - b)| \in [\underline{y}^i, \overline{y}^i],$$

$$\tilde{x}^i + M|R(A\tilde{x}^i + B|\tilde{x}^i| - b)| \in [\underline{z}^i, \overline{z}^i],$$

where $\underline{y}^i, \overline{y}^i, \underline{z}^i, \overline{z}^i$ are floating-point vectors; then by Theorem 3 we have the verified enclosure

$$x^* \in [\underline{y}^i, \overline{z}^i]$$

for each $i = 0, 1, 2, \dots$. Now define

$$\boldsymbol{x}^0 = [\underline{y}^0, \overline{z}^0]$$

and

$$\boldsymbol{x}^{i+1} = [\underline{y}^{i+1}, \overline{z}^{i+1}] \cap \boldsymbol{x}^i \qquad (i = 0, 1, 2, \dots),$$

then

$$\boldsymbol{x}^0 \supseteq \boldsymbol{x}^1 \supseteq \dots \supseteq \boldsymbol{x}^i \supseteq \dots \ni x^*$$

so that we get a nested sequence of floating-point interval vectors each of whom is verified to contain the solution x^*.

4 The inequality (1.2)

Let us write the condition (1.2), as before, in the form

$$M(I - G) \geq I, \tag{4.1}$$

$$M \geq 0, \tag{4.2}$$

where

$$G = |I - RA| + |RB|.$$

In this section we shall be interested in finding M and R satisfying (4.1), (4.2). Obviously, such M and R do not exist always because their existence implies unique solvability of (1.1) (Theorem 1). We show how M and R satisfying (4.1), (4.2) can be found.

Theorem 4. *If (assuming both inverses to exist)*

$$(I - |A^{-1}B|)^{-1} \geq 0,$$

then

$$R = A^{-1}, \tag{4.3}$$

$$M = (I - |A^{-1}B|)^{-1} \tag{4.4}$$

satisfy (4.1), (4.2).

Proof. Indeed, $M \geq 0$ by assumption, and $M(I - |I - RA| - |RB|) = M(I - |A^{-1}B|) = I$. ■

This gives us the clue: in practical computations choose R and M as the *computed* values of (4.3), (4.4).

5 Condensed form of conditions (4.1), (4.2)

The two conditions (4.1), (4.2) can be merged into a single "condensed" condition

$$M \geq I + |M|G. \tag{5.1}$$

In fact, if (5.1) holds, then $M \geq 0$ (because $G \geq 0$), hence $|M| = M$ and (5.1) turns into (4.1). Conversely, if (4.1), (4.2) hold, then $M = |M|$ and the condition (4.1), if written as $M \geq I + MG$, becomes (5.1).

This, of course, is only a technical trick, which, however, spares us of necessity of emphasizing nonnegativity of M whenever mentioning the condition (4.1).

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