Diagonally Singularizable Matrices

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Abstract

A square matrix A is called diagonally singularizable if $|A - S| \leq I$ holds for some singular matrix S (I is the identity matrix). The paper brings several necessary and/or sufficient conditions for diagonal singularizability and demonstrates another specific features, namely existence of diagonal-singularizability-preserving operations and a theorem of symmetric alternative.

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1. Introduction

In [1], the authors introduced the following concept: a matrix $A \in \mathbb{R}^{n\times n}$ is called *diagonally singularizable* if there exists a singular matrix S satisfying

$$|A - S| \leq I,$$

where I denotes the identity matrix and absolute value and inequality are understood entrywise. This term was used in [1] for formulating a nontrivial assertion (see Theorem 5 below): for each nonsingular matrix A, either A or A^{-1} is diagonally singularizable. It was just this remarkable property that prompted this author to investigate the concept of diagonal singularizability in more detail, thus giving rise to the present paper which brings several necessary and/or sufficient conditions for diagonal singularizability

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** Dedicated to Professor Ilja Černý on the occasion of his 90th birthday.
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and demonstrates another specific features, namely existence of diagonal-singularizability-preserving operations and a theorem of symmetric alternative.

We use the following notation. $\varrho(A)$ stands for the spectral radius of A and $\lambda_{\min}(A)$ denotes the minimum eigenvalue of a symmetric matrix A. Let us recall that by the Courant-Fischer theorem [2],

$$\lambda_{\min}(A) = \min_{x \neq 0} \frac{x^T A x}{x^T x}.$$

Continuity of the minimum eigenvalue follows from the Wielandt-Hoffman theorem (see [2]):

$$|\lambda_{\min}(A) - \lambda_{\min}(B)| \leq \|A - B\|_F$$

holds for any two symmetric matrices $A, B \in \mathbb{R}^{n \times n}$, where we use the Frobenius matrix norm $\|C\|_F = (\sum_{ij} c_{ij}^2)^{1/2}$. An interval matrix is a set of matrices of the form

$$[A - D,\, A + D] = \{\, C \mid |A - C| \leq D \,\}$$

with $D \geq 0$; it is called singular if it contains a singular matrix. For a $t \in \mathbb{R}^n$, T_t denotes the diagonal matrix with diagonal vector t. $\{-1, 1\}^n$ is the set of all ± 1-vectors in $\mathbb{R}^n$ (there are 2^n of them). Let us note that $|AB| \leq |A||B|$ whenever the matrices A, B can be multiplied.

2. Necessary and sufficient conditions

First, we have several necessary and sufficient conditions for diagonal singularizability.

Theorem 1. *For a matrix $A \in \mathbb{R}^{n \times n}$, the following assertions are equivalent:*

(i) *A is diagonally singularizable,*

(ii) *$[A - I,\, A + I]$ is singular,*

(iii) *$|Ax| \leq |x|$ for some $x \neq 0$,*

(iv) *$\det(A)\det(A - T_y) \leq 0$ for some $y \in \{-1, 1\}^n$,*

(v) *$A - \tau T_y$ is singular for some $\tau \in [0, 1]$ and $y \in \{-1, 1\}^n$,*

(vi) $|Ax| = \tau|x|$ *for some* $\tau \in [0,1]$ *and* $x \neq 0$.

PROOF. Let $A \in \mathbb{R}^{n\times n}$. We shall prove that i$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(vi) $\Rightarrow$(i).

(i)$\Rightarrow$(ii): If $|A - S| \leq I$ for some singular S, then $S \in [A-I,\ A+I]$, hence $[A-I,\ A+I]$ is singular.

(ii)$\Rightarrow$(iii): If $[A-I,\ A+I]$ contains a singular matrix S, then $Sx = 0$ for some $x \neq 0$ which implies that $|Ax| = |(A-S)x| \leq |A-S||x| \leq |x|$.

(iii)$\Rightarrow$(iv): Let $|Ax| \leq |x|$ for some $x \neq 0$. Put $t_i = (Ax)_i/x_i$ if $x_i \neq 0$ and $t_i = 1$ otherwise ($i = 1,\ldots,n$), then $t_i \in [-1,1]$ and $(Ax)_i = t_i x_i$ for each i which can be written as $(A - T_t)x = 0$ where $t = (t_i)$, implying

$$\det(A - T_t) = 0. \tag{1}$$

Now define a function f by

$$f(s) = \det(A)\det(A - T_s), \quad s \in \mathbb{R}^n. \tag{2}$$

We shall construct by induction numbers $y_i \in \{-1,1\}$, $i = 1,\ldots,n$, such that

$$f(y_1,\ldots,y_i,t_{i+1},\ldots,t_n) \leq 0 \tag{3}$$

will hold for $i = 0,\ldots,n$. For $i = 0$ this follows from (1). Thus assume that numbers $y_1,\ldots,y_{i-1}$ satisfying

$$f(y_1,\ldots,y_{i-1},t_i,t_{i+i},\ldots,t_n) \leq 0 \tag{4}$$

have been already constructed for some i, $1 \leq i \leq n$. Define a function of one variable

$$\phi_i(\sigma) = f(y_1,\ldots,y_{i-1},\sigma,t_{i+1},\ldots,t_n)$$

and construct y_i as follows: let $y_i = -1$ if

$$\phi_i(-1) \leq \phi_i(1) \tag{5}$$

and set $y_i = 1$ otherwise. It follows from the Laplace expansion of the second determinant in (2) along the ith row that $\phi_i(\sigma)$ is linear in σ. Thus if (5) holds, then $\phi_i(\sigma)$ is nondecreasing and because $t_i \in [-1,1]$ (see the definition of t_i above), we have

$$\phi_i(y_i) = \phi_i(-1) \leq \phi_i(t_i) \leq 0$$

by the induction assumption (4); if

$$\phi_i(-1) > \phi_i(1)$$

holds, then $\phi_i(\sigma)$ is decreasing and we have

$$\phi_i(y_i) = \phi_i(1) \le \phi_i(t_i) \le 0$$

so that in both cases (3) holds which concludes the proof by induction. Now from (3) for $i = n$ we obtain that $f(y) \le 0$ which was to be proved.

(iv)$\Rightarrow$(v): Let $\det(A)\det(A - T_y) \le 0$ for some $y \in \{-1,1\}^n$. Define a real function g by

$$g(t) = \det(A - tT_y), \quad t \in [0,1].$$

Then g is continuous in $[0,1]$ and $g(0)g(1) = \det(A)\det(A - T_y) \le 0$ which in the light of the intermediate value theorem means that $g(\tau) = 0$ for some $\tau \in [0,1]$, hence $A - \tau T_y$ is singular.

(v)$\Rightarrow$(vi): If $A - \tau T_y$ is singular for some $\tau \in [0,1]$ and $y \in \{-1,1\}^n$, then $(A - \tau T_y)x = 0$ for some $x \ne 0$ which implies $|Ax| = \tau|x|$.

(vi)$\Rightarrow$(i): Let $|Ax| = \tau|x|$ for some $\tau \in [0,1]$ and $x \ne 0$. Define $y, z \in \{-1,1\}^n$ by $y_i = 1$ if $(Ax)_i \ge 0$ and $y_i = -1$ otherwise, and $z_i = 1$ if $x_i \ge 0$ and $z_i = -1$ otherwise ($i = 1, \dots, n$), then $T_yAx = \tau T_z x$ implying $(A - \tau T_y T_z)x = 0$ which shows that the matrix $S = A - \tau T_y T_z$ is singular and satisfies $|A - S| = |\tau T_y T_z| \le I$, hence A is diagonally singularizable.

3. Sufficient conditions

Next we have two sufficient conditions for diagonal singularizability and its negation.

Theorem 2. *If A is nonsingular and*

$$\max_j |A^{-1}|_{jj} \ge 1$$

holds, then A is diagonally singularizable.

PROOF. Take a k for which $|A^{-1}|_{kk} \ge 1$. Then

$$|A^{-1}e_k| = |A^{-1}|e_k \ge e_k = |e_k|, \tag{6}$$

where e_k is the kth column of the identity matrix I. Put $x = A^{-1}e_k$, then $x \ne 0$ and from (6) we obtain

$$|Ax| \le |x|,$$

which by Theorem 1, (iii) means that A is diagonally singularizable.

Theorem 3. *If A is nonsingular and*

$$\varrho(|A^{-1}|) < 1 \tag{7}$$

holds, then A is not diagonally singularizable.

PROOF. Let us recall that (7) implies $(I - |A^{-1}|)^{-1} \geq 0$ (Horn and Johnson [3]). Assume to the contrary that A is diagonally singularizable, so that $|Ax| \leq |x|$ for some $x \neq 0$ (Theorem 1, (iii)). Put $x' = Ax$, then $x' \neq 0$ and it satisfies $|x'| \leq |A^{-1}x'|$ which implies

$$|x'| \leq |A^{-1}||x'|,$$

hence

$$(I - |A^{-1}|)|x'| \leq 0,$$

and premultiplying this inequality by the nonnegative matrix $(I - |A^{-1}|)^{-1}$ yields $|x'| \leq 0$, hence $x' = 0$ which contradicts the previously mentioned fact that $x' \neq 0$.

4. Operations preserving diagonal singularizability

In this section we show that three well-known matrix operations preserve diagonal singularizability.

Theorem 4. *If a square matrix A is diagonally singularizable, then so are A^T, $(A^TA)^{2^k}$ and $(AA^T)^{2^k}$ for $k = 0, 1, 2, \ldots$.*

PROOF. (a) If A is diagonally singularizable, then from $|A - S| \leq I$ for some singular S it follows that $|A^T - S^T| \leq I$ where S^T is again singular, hence A^T is diagonally singularizable.

(b) We shall first prove diagonal singularizability of A^TA. By Theorem 1, (iii) a diagonally singularizable A satisfies $|Ax| \leq |x|$ for some $x \neq 0$. Then we have

$$x^TA^TAx = (Ax)^T(Ax) \leq |Ax|^T|Ax| \leq |x|^T|x| = x^Tx,$$

hence

$$x^T(A^TA - I)x \leq 0$$

which in view of symmetry of $A^TA - I$ implies that

$$\lambda_{\min}(A^TA - I) = \min_{x' \neq 0} \frac{x'^T(A^TA - I)x'}{x'^Tx'} \leq \frac{x^T(A^TA - I)x}{x^Tx} \leq 0.$$

Now define a real function h by

$$h(t) = \lambda_{\min}(A^T A - tI), \quad t \in [0,1].$$

It is well defined because $A^T A - tI$ is symmetric for each $t \in [0,1]$, and the above reasoning implies that $h(1) \leq 0$. Next, we have that $h(0) = \lambda_{\min}(A^T A) \geq 0$ because $A^T A$ is symmetric positive semidefinite, and h is continuous in $[0,1]$ since for each $t_1, t_2 \in [0,1]$ we have by the Wielandt-Hoffman theorem that

$$|h(t_1) - h(t_2)| \leq \|(t_1 - t_2)I\|_F = n^{1/2}|t_1 - t_2|.$$

Hence the intermediate value theorem implies existence of a $\tau \in [0,1]$ such that $h(\tau) = 0$ which gives that $\lambda_{\min}(A^T A - \tau I) = 0$. Thus $A^T A - \tau I$ is singular and Theorem 1, (v) (with $y = e$, the vector of all ones) proves $A^T A$ to be diagonally singularizable.

(c) Next we prove by induction on k that $(A^T A)^{2^k}$ is diagonally singularizable for $k = 0, 1, 2, \ldots$. The case of $k = 0$ has been proved in part (b). Thus assume that $(A^T A)^{2^k}$ is diagonally singularizable for some $k \geq 0$. Then, again by part (b), $((A^T A)^{2^k})^T (A^T A)^{2^k} = ((A^T A)^{2^k})^2 = (A^T A)^{2^{k+1}}$ is diagonally singularizable which concludes the proof by induction.

(d) If we apply the previous result to A^T, we obtain that $(AA^T)^{2^k}$ is diagonally singularizable for $k = 0, 1, 2, \ldots$.

5. Symmetric alternative

The situation with matrix inverse is different. The following theorem was proved in [1].

Theorem 5. *For each nonsingular matrix A at least one of the matrices A, A^{-1} is diagonally singularizable.*

We derive two consequences of this result. The minimum/maximum of two matrices is understood entrywise.

Theorem 6. *For each nonsingular matrix A the interval matrix*

$$[\min\{A, A^{-1}\} - I, \max\{A, A^{-1}\} + I] \tag{8}$$

is singular.

PROOF. By Theorem 5 at least one of the interval matrices $[A - I,\ A + I]$, $[A^{-1} - I,\ A^{-1} + I]$ is singular, therefore (8), the minimal (w.r.t. inclusion) interval matrix enclosing both these interval matrices, contains a singular matrix.

Finally, we have this "symmetric alternative".

Theorem 7. *For each square matrix A at least one of the inequalities*

$$|Ax| \leq |x|, \tag{9}$$
$$|x| \leq |Ax| \tag{10}$$

has a nontrivial solution.

PROOF. If A is singular, then a nontrivial solution to $Ax = 0$ solves (9). If A is nonsingular and (9) does not possess a nontrivial solution, then A is not diagonally singularizable, hence the inverse A^{-1} is diagonally singularizable by Theorem 5 so that $|A^{-1}x'| \leq |x'|$ has a nontrivial solution x'; then $x = A^{-1}x'$ is a nontrivial solution to (10).

6. Checking diagonal singularizability

How to check diagonal singularizability? If none of the sufficient conditions of Theorems 2 and 3 works, it seems the best option to check the interval matrix $[A-I,\ A+I]$ for singularity. This can be done by MATLAB program **regising.m** which is freely available at `http://uivtx.cs.cas.cz/~rohn/other/regising.m`. For our purposes it should be invoked by
`S=regising(A,eye(size(A)))`.
If S is nonempty, then S is a singular matrix satisfying $|A - S| \leq I$, i.e., A is diagonally singularizable; if it is empty, then no such matrix exists and A is not diagonally singularizable.

For the following randomly generated example both the sufficient conditions fail and we must resort to use of the general algorithm.

```
>> rand('state',1);  A=2*rand(5,5)-1

A =

    0.9056   -0.1144    0.7972    0.8205    0.5374
    0.4081    0.6736   -0.1420    0.0506   -0.8810
    0.9078    0.0374   -0.6009   -0.3863    0.2542
    0.1963   -0.9556   -0.3938   -0.9311   -0.4696
    0.6815   -0.2482    0.0766    0.4307   -0.3753
```

The matrix is nonsingular as shown by

```
>> rank(A)

ans =

     5
```

Now we use

```
>> S=regising(A,eye(size(A)))
```

to obtain

```
S =

    1.0052   -0.1144    0.7972    0.8205    0.5374
    0.4081    1.6736   -0.1420    0.0506   -0.8810
    0.9078    0.0374    0.3991   -0.3863    0.2542
    0.1963   -0.9556   -0.3938   -1.9311   -0.4696
    0.6815   -0.2482    0.0766    0.4307   -1.3753
```

Notice that indeed the off-diagonal entries of both matrices are the same, and the diagonal ones have been shifted by an amount of at most 1 each. We may check singularity of S by computing its rank.

```
>> rank(S)

ans =

     4
```

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