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A General Method for Enclosing Solutions of Interval Linear Equations

Jiri Rohn

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Abstract We describe a general method for enclosing the solution set of a system of interval linear equations. We present a general theorem and an algorithm in a MATLAB-style code.

Keywords Interval linear equations · solution set · enclosure · absolute value inequality.

1 Introduction

In this paper we describe a general method for enclosing the solution set of a system of interval linear equations. We present a general theorem (Theorem 3) and an algorithm in a MATLAB-style code (Fig. 1). We call the result a "method", not an "algorithm", because it involves solving absolute value matrix inequalities; different solutions yield different enclosures. We plan to elaborate on this issue in a forthcoming paper.

The problem of enclosing the solution set of systems of interval linear equations arises when solving global optimization problems or rigorously locating all solutions to nonlinear systems, see, e.g., Kearfott [5]. Some related articles can be found in [3].

2 Notations

We use the following notations. Matrix inequalities, as $A \le B$ or $A < B$, are understood componentwise. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. The same notations also apply to vectors that are considered one-column matrices. I is the unit matrix, e_j is the jth column of I, and $e = (1, \dots, 1)^T$ is the vector of all ones.

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J. Rohn
Institute of Computer Science, Czech Academy of Sciences, Prague, Czech Republic
E-mail: rohn@cs.cas.cz

J. Rohn
School of Business Administration, Anglo-American University, Prague, Czech Republic

$Y_n = \{y \mid |y| = e\} = \{-1, 1\}^n$ is the set of all ± 1-vectors in $\mathbb{R}^n$, so that its cardinality is 2^n. Vectors $y, z \in Y_n$ are called *adjacent* if they differ in exactly one entry. Obviously, $y, z \in Y_n$ are adjacent if and only if $y = z - 2z_j e_j$ for some j. For each $x \in \mathbb{R}^n$ we define its sign vector $\operatorname{sgn}(x)$ by

$$(\operatorname{sgn}(x))_i = \begin{cases} 1 \text{ if } x_i \geq 0, \\ -1 \text{ if } x_i < 0 \end{cases} \qquad (i = 1, \ldots, n),$$

so that $\operatorname{sgn}(x) \in Y_n$. For each $z \in \mathbb{R}^n$ we denote

$$T_z = \operatorname{diag}(z_1, \ldots, z_n) = \begin{pmatrix} z_1 & 0 & \ldots & 0 \\ 0 & z_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & z_n \end{pmatrix},$$

and $\mathbb{R}^n_z = \{x \mid T_z x \geq 0\}$ is the orthant prescribed by the ± 1-vector $z \in Y_n$. An interval matrix is a set of matrices

$$\mathbf{A} = \{ A \mid |A - A_c| \leq \Delta \} = [A_c - \Delta, A_c + \Delta],$$

and an interval vector is a one-column interval matrix

$$\mathbf{b} = \{ b \mid |b - b_c| \leq \delta \} = [b_c - \delta, b_c + \delta].$$

3 The problem

Given an $n \times n$ interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ and an interval n-vector $\mathbf{b} = [b_c - \delta, b_c + \delta]$, the solution set of the system of interval linear equations $\mathbf{A}x = \mathbf{b}$ is defined as

$$\mathbf{X}(\mathbf{A}, \mathbf{b}) = \{ x \mid Ax = b \text{ for some } A \in \mathbf{A},\ b \in \mathbf{b} \}.$$

The Oettli-Prager theorem [6] asserts that the solution set is described by

$$\mathbf{X}(\mathbf{A}, \mathbf{b}) = \{ x \mid |A_c x - b_c| \leq \Delta |x| + \delta \}.$$

If $\mathbf{A}$ is regular, then $\mathbf{X}(\mathbf{A}, \mathbf{b})$ is compact and connected (Beeck [1]); if $\mathbf{A}$ is singular, then *each* component of $\mathbf{X}(\mathbf{A}, \mathbf{b})$ is unbounded (Jansson [4]). The solution set is generally of a complicated nonconvex structure. In practical computations, therefore, we look for an *enclosure* of it, i.e., for an interval vector $\mathbf{x}$ satisfying

$$\mathbf{X}(\mathbf{A}, \mathbf{b}) \subseteq \mathbf{x}.$$

The present text is dedicated to the problem of finding such an $\mathbf{x}$ under general circumstances when regularity/singularity of $\mathbf{A}$ is not known in advance (and is verified on the way). The text owes much to Christian Jansson's ideas in [4].

4 The results

The core of our method consists in specifying a possibly small subset Z of Y_n such that

$$\mathbf{X}(\mathbf{A},\mathbf{b}) \subseteq \bigcup_{z\in Z} \mathbb{R}^n_z.$$

In the first theorem such a set Z is described recursively ((a), (c) below) in terms of the solution set only.

Theorem 1 *Let* $\mathbf{A}$ *be an* $n\times n$ *interval matrix,* $\mathbf{b}$ *an interval* n*-vector, and let* Z *be a subset of* Y_n *having the following properties:*

(a) $\operatorname{sgn}(x_0) \in Z$ *for some* $x_0 \in \mathbf{X}(\mathbf{A},\mathbf{b})$*,*
(b) $\mathbf{X}(\mathbf{A},\mathbf{b}) \cap \mathbb{R}^n_z$ *is bounded for each* $z \in Z$*,*
(c) if z, y *are adjacent,* $z \in Z$*,* $y \in Y_n$*, and* $\mathbf{X}(\mathbf{A},\mathbf{b}) \cap \mathbb{R}^n_z \cap \mathbb{R}^n_y \neq \emptyset$*, then* $y \in Z$*.*

Then $\mathbf{A}$ *is regular and*

$$\mathbf{X}(\mathbf{A},\mathbf{b}) \subseteq \bigcup_{z\in Z} \mathbb{R}^n_z \tag{1}$$

holds.

Proof For brevity, denote $X = \mathbf{X}(\mathbf{A},\mathbf{b})$. Let X_0 be the component of X (i.e. a nonempty connected subset of X maximal with respect to inclusion) containing x_0. We shall prove that

$$X_0 \subseteq \bigcup_{z\in Z} \mathbb{R}^n_z \tag{2}$$

holds. Assume to the contrary that it is not so, so that there exists an $x_1 \in X_0$ such that

$$x_1 \notin \bigcup_{z\in Z} \mathbb{R}^n_z.$$

Since X_0 is connected, there exists a continuous mapping $\varphi : [0,1] \to X_0$ with $\varphi(0) = x_0$ and $\varphi(1) = x_1$. Let

$$\tau = \sup\{\, t \mid \varphi(t) \in \bigcup_{z\in Z} \mathbb{R}^n_z \,\},$$

and put $x^* = \varphi(\tau)$. Then $x^* \in \bigcup_{z\in Z} \mathbb{R}^n_z$ because φ is continuous and $\bigcup_{z\in Z} \mathbb{R}^n_z$ is closed, say $x^* \in \mathbb{R}^n_{z'}$, $z' \in Z$, hence $x^* \neq x_1$ and $\tau < 1$. Put $\varepsilon = 1 - \tau$ and consider the sequence

$$\{\varphi(\tau + \varepsilon/j)\}_{j=1}^{\infty}.$$

Since

$$\varphi(\tau + \varepsilon/j) \in \bigcup_{z\notin Z} \mathbb{R}^n_z$$

for each j and since the set $\{\, z \in Y_n \mid z \notin Z \,\}$ is finite, there exists a $z'' \notin Z$ such that $\varphi(\tau + \varepsilon/j) \in \mathbb{R}^n_{z''}$ for infinitely many j. Taking the limit along this subsequence, we get that $x^* \in \mathbb{R}^n_{z''}$ because $\mathbb{R}^n_{z''}$ is closed. Thus we have that

$$x^* \in \mathbb{R}^n_{z'} \cap \mathbb{R}^n_{z''}$$

where $z' \in Z$ and $z'' \notin Z$, so that $z' \neq z''$. Put

$$I = \{\, i \mid z'_i \neq z''_i \,\} = \{i_1, \ldots, i_m\},$$

then

$$x^*_i = 0$$

for each $i \in I$, and define vectors $z^0, z^1, \ldots, z^m \in Y_n$ by induction as follows:

$$z^0 = z'$$

and

$$z^j := z^{j-1},\ z^j_{i_j} := -z^j_{i_j}$$

for $j = 1, \ldots, m$. Then $z^0 \in Z$ and by induction for each $j = 1, \ldots, m$, z^{j-1} and z^j are adjacent, $z^{j-1} \in Z$ and $x^* \in \mathbb{R}^n_{z^{j-1}} \cap \mathbb{R}^n_{z^j}$, $x^* \in X_0 \subseteq X$, hence $z^j \in Z$ by assumption (c). Thus, by induction, $z^j \in Z$ for each $j = 0, \ldots, m$. In particular, $z'' = z^m \in Z$, which contradicts the previously established fact that $z'' \notin Z$. This contradiction finally proves that (2) holds.

Now, (2) implies that

$$X_0 \subseteq \bigcup_{z \in Z} (X_0 \cap \mathbb{R}^n_z) \subseteq \bigcup_{z \in Z} (X \cap \mathbb{R}^n_z),$$

hence the component X_0 is bounded by assumption (b). If $\mathbf{A}$ were singular, then, by Jansson's result in [4], *each* component of X would be unbounded. Since X_0 is bounded, this implies that $\mathbf{A}$ is regular and therefore X is connected (Beeck [1]); this means that $X_0 = X$, and (2) implies (1). □

In the second theorem we further assume existence of an enclosure of each nonempty set $\mathbf{X}(\mathbf{A}, \mathbf{b}) \cap \mathbb{R}^n_z$, $z \in Z$ (without specifying how such an enclosure should be found).

Theorem 2 *Let* $\mathbf{A}$ *be an* $n \times n$ *interval matrix,* $\mathbf{b}$ *an interval* n*-vector, and let* Z *be a subset of* Y_n *having the following properties:*

(a') $\operatorname{sgn}(x_0) \in Z$ *for some* $x_0 \in \mathbf{X}(\mathbf{A}, \mathbf{b})$*,*
(b') for each $z \in Z$ *such that* $\mathbf{X}(\mathbf{A}, \mathbf{b}) \cap \mathbb{R}^n_z \neq \emptyset$ *there exists an interval vector* $[\underline{x}_z, \overline{x}_z]$ *satisfying* $\mathbf{X}(\mathbf{A}, \mathbf{b}) \cap \mathbb{R}^n_z \subseteq [\underline{x}_z, \overline{x}_z]$*,*
(c') if $z \in Z$*,* $\mathbf{X}(\mathbf{A}, \mathbf{b}) \cap \mathbb{R}^n_z \neq \emptyset$*, and* $(\underline{x}_z)_j(\overline{x}_z)_j \leq 0$ *for some* j*, then* $z - 2z_je_j \in Z$*.*

Then $\mathbf{A}$ *is regular and*

$$\mathbf{X}(\mathbf{A}, \mathbf{b}) \subseteq \bigcup_{z \in Z_0} [\underline{x}_z, \overline{x}_z]$$

holds, where

$$Z_0 = \{\, z \in Z \mid \mathbf{X}(\mathbf{A}, \mathbf{b}) \cap \mathbb{R}^n_z \neq \emptyset \,\}.$$

Proof We shall prove that assumptions (a'), (b'), (c') imply validity of the assumptions (a), (b), (c) of Theorem 1. (a') and (a) are the same, and (b') clearly implies (b). To prove (c), let z, y be adjacent, $z \in Z$, $y \in Y_n$, and let $\mathbf{X}(\mathbf{A}, \mathbf{b}) \cap \mathbb{R}^n_z \cap \mathbb{R}^n_y \neq \emptyset$. Then there exists a j such that $z_k = y_k$ for each $k \neq j$ and $z_j = -y_j$, and there exists an $x \in \mathbf{X}(\mathbf{A}, \mathbf{b}) \cap \mathbb{R}^n_z \cap \mathbb{R}^n_y$ which clearly satisfies $x_j = 0$, hence, by (b'),

$$(\underline{x}_z)_j \leq 0 \leq (\overline{x}_z)_j$$

and therefore

$$(\underline{x}_z)_j(\overline{x}_z)_j \le 0,$$

hence $y = z - 2z_je_j \in Z$ by (c'), which proves (c). Thus the assumptions of Theorem 1 are met and we obtain that $\mathbf{A}$ is regular and

$$\mathbf{X}(\mathbf{A}, \mathbf{b}) \subseteq \bigcup_{z\in Z} \mathbb{R}^n_z$$

holds, which in conjunction with assumption (b') and the definition of Z_0 gives

$$\mathbf{X}(\mathbf{A}, \mathbf{b}) \subseteq \bigcup_{z\in Z} (\mathbf{X}(\mathbf{A}, \mathbf{b}) \cap \mathbb{R}^n_z) = \bigcup_{z\in Z_0} (\mathbf{X}(\mathbf{A}, \mathbf{b}) \cap \mathbb{R}^n_z) \subseteq \bigcup_{z\in Z_0} [\underline{x}_z, \overline{x}_z].$$

□

Finally, in the third theorem we specify a way how to enclose the sets $\mathbf{X}(\mathbf{A}, \mathbf{b}) \cap \mathbb{R}^n_z \ne \emptyset$, $z \in Z$, via solutions of certain nonlinear matrix inequalities. Thus, this theorem describes a construction of a set Z as well as a construction of orthantwise enclosures.

Theorem 3 *Let $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ be an $n \times n$ interval matrix, $\mathbf{b} = [b_c - \delta, b_c + \delta]$ an interval n-vector, and let Z be a subset of Y_n having the following properties:*

(a") $\operatorname{sgn}(x_0) \in Z$ *for some* $x_0 \in \mathbf{X}(\mathbf{A}, \mathbf{b})$,
(b") for each $z \in Z$ the inequalities

$$(QA_c - I)T_z \ \ \ge |Q|\Delta, \tag{3}$$
$$(QA_c - I)T_{-z} \ge |Q|\Delta \tag{4}$$

have matrix solutions Q_z and Q_{-z}, respectively; denote

$$\overline{x}_z = Q_zb_c + |Q_z|\delta, \tag{5}$$
$$\underline{x}_z = Q_{-z}b_c - |Q_{-z}|\delta, \tag{6}$$

(c") if $z \in Z$, $\underline{x}_z \le \overline{x}_z$, and $(\underline{x}_z)_j(\overline{x}_z)_j \le 0$ for some j, then $z - 2z_je_j \in Z$.

Then $\mathbf{A}$ is regular and

$$\mathbf{X}(\mathbf{A}, \mathbf{b}) \subseteq [\min_{z\in Z_1} \underline{x}_z, \max_{z\in Z_1} \overline{x}_z] \tag{7}$$

holds, where

$$Z_1 = \{\, z \in Z \mid \underline{x}_z \le \overline{x}_z \,\}. \tag{8}$$

Proof Let $z \in Z$, $\mathbf{X}(\mathbf{A}, \mathbf{b}) \cap \mathbb{R}^n_z \ne \emptyset$, and let Q_z solve (3), so that it satisfies

$$T_z \le Q_zA_cT_z - |Q_z|\Delta. \tag{9}$$

Then for each $x \in \mathbf{X}(\mathbf{A}, \mathbf{b}) \cap \mathbb{R}^n_z$ we have $T_zx = |x|$, $x = T_z|x|$, and

$$|A_cx - b_c| \le \Delta|x| + \delta \tag{10}$$

by the Oettli-Prager theorem ([6], in the current form in [2]). First postmultiplying (9) by $|x|$ and later premultiplying (10) by $|Q_z|$, we obtain

$$\begin{aligned} x &= T_z|x| \le Q_z A_c T_z|x| - |Q_z|\Delta|x| \\ &= Q_z A_c x - |Q_z|\Delta|x| \\ &= Q_z(A_c x - b_c) + Q_z b_c - |Q_z|\Delta|x| \\ &\le |Q_z(A_c x - b_c)| + Q_z b_c - |Q_z|\Delta|x| \\ &\le |Q_z||A_c x - b_c| + Q_z b_c - |Q_z|\Delta|x| \\ &\le |Q_z|(\Delta|x| + \delta) + Q_z b_c - |Q_z|\Delta|x| \\ &= Q_z b_c + |Q_z|\delta = \overline{x}_z. \end{aligned}$$

Similarly, since $T_{-z} = -T_z$, the inequality (4) can be written as

$$T_z \ge Q_{-z} A_c T_z + |Q_{-z}|\Delta,$$

and we have

$$\begin{aligned} x &= T_z|x| \ge Q_{-z} A_c T_z|x| + |Q_{-z}|\Delta|x| \\ &= Q_{-z} A_c x + |Q_{-z}|\Delta|x| \\ &= Q_{-z}(A_c x - b_c) + Q_{-z} b_c + |Q_{-z}|\Delta|x| \\ &\ge -|Q_{-z}(A_c x - b_c)| + Q_{-z} b_c + |Q_{-z}|\Delta|x| \\ &\ge -|Q_{-z}||A_c x - b_c| + Q_{-z} b_c + |Q_{-z}|\Delta|x| \\ &\ge -|Q_{-z}|(\Delta|x| + \delta) + Q_{-z} b_c + |Q_{-z}|\Delta|x| \\ &= Q_{-z} b_c - |Q_{-z}|\delta = \underline{x}_z, \end{aligned}$$

In this way we have proved that

$$\mathbf{X}(\mathbf{A}, \mathbf{b}) \cap \mathbb{R}^n_z \subseteq [\underline{x}_z, \overline{x}_z].$$

Thus the assumptions (a')-(c') of Theorem 2 are met and the result follows from it since

$$Z_0 = \{ z \in Z \mid \mathbf{X}(\mathbf{A}, \mathbf{b}) \cap \mathbb{R}^n_z \ne \emptyset \} \subseteq \{ z \in Z \mid \underline{x}_z \le \overline{x}_z \} = Z_1.$$

□

5 A general method

Theorem 3 has been translated into a MATLAB-style code in Fig. 1. The text is self-explanatory as the same notations are used. If the algorithm terminates successfully in line (25), then $D = Z_1$. The following result is immediate:

Theorem 4 *For each $n \times n$ interval matrix $\mathbf{A}$ and for each interval n-vector $\mathbf{b}$ the algorithm (Fig. 1) in a finite number of steps either computes an enclosure X of the solution set of the interval linear system $\mathbf{A}x = \mathbf{b}$, or fails (produces an empty output).*

```
function X = genmeth (A, b)
% Computes an enclosure X of the solution set
% of Ax = b, or produces an empty output.
 if A_c is singular, X = []; return, end
 x_c = A_c^{-1} b_c; z = sgn(x_c); x_lower = x_c; x_upper = x_c;
 Z = {z}; D = ∅;
 while Z ≠ ∅
    select z ∈ Z; Z = Z − {z}; D = D ∪ {z};
    find a solution Q_z of (QA_c − I)T_z ≥ |Q|Δ;
    if Q_z not found, X = []; return, end
    find a solution Q_{−z} of (QA_c − I)T_{−z} ≥ |Q|Δ;
    if Q_{−z} not found, X = []; return, end
    x_upper_z = Q_z b_c + |Q_z|δ;
    x_lower_z = Q_{−z} b_c − |Q_{−z}|δ;
    if x_lower_z ≤ x_upper_z
       x_lower = min(x_lower, x_lower_z); x_upper = max(x_upper, x_upper_z);
       for j = 1 : n
          z' = z; z'_j = −z'_j;
          if ((x_lower_z)_j (x_upper_z)_j ≤ 0 and z' ∉ Z ∪ D)
             Z = Z ∪ {z'};
          end
       end
    end
 end
 X = [x_lower, x_upper];
```

Fig. 1 A general method for computing enclosures.

6 The role of Theorem 3

Theorem 3, published here with some delay, has been used in the freely available verification software package VERSOFT [11] as the main tool behind the function VERINTERVALHULL.M [8] for computing the interval hull of the solution set of a system of interval linear equations (see [12]). This function, in turn, is then called by VERSOFT functions VERREGSING.M [10], VERPOSDEF.M [9], and VERBASINT-NPPROB.M [7]. All these functions use not-a-priori-exponential algorithms for solving NP-hard problems. This is due to use of the subset Z_1 introduced in (8) instead of the whole of Y_n. This explains that Theorem 3 plays in fact a more important role than this short paper might suggest.

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