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A Note on Generating P-Matrices

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Abstract We prove that for any $A, B \in \mathbb{R}^{n\times n}$ such that each matrix S satisfying $\min(A,B) \le S \le \max(A,B)$ is nonsingular, all four matrices $A^{-1}B$, AB^{-1}, $B^{-1}A$ and BA^{-1} are P-matrices. A practical method for generating P-matrices is drawn from this result.

Keywords P-matrix · interval matrix.

1 Introduction

A square matrix is called a P-matrix if all its principal minors are positive. Fiedler and Pták in their now famous paper [3] proved that A is a P-matrix if and only if no $x \neq 0$ satisfies $x \circ Ax \le 0$, where "$\circ$" stands for the Hadamard product. This nice and far-reaching theoretical result does not show, however, how to verify the P-property in practical computations. And indeed, Coxson proved in [2] that the problem of checking whether a given square matrix is a P-matrix is co-NP-complete. The special case of a symmetric A can be handled in polynomial time because such an A is a P-matrix if and only if it is positive definite [3], but the general case remains difficult.

The author was confronted with the problem of constructing nontrivial nonsymmetric P-matrices while working on the MATLAB/INTLAB file VERPMAT.M [7] based on a not-a-priori exponential algorithm for checking the P-property (which is going to be described elsewhere). An answer is given in the two theorems below, of which the first one is more general and the second one is more practically oriented.

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2 The results

Both the results below show how a P-matrix can be constructed from two matrices satisfying certain conditions.

Theorem 1 *For each* $A, B \in \mathbb{R}^{n\times n}$ *such that each matrix* S *satisfying*

$$\min(A, B) \leq S \leq \max(A, B)$$

is nonsingular, all four matrices $A^{-1}B$*,* AB^{-1}*,* $B^{-1}A$ *and* BA^{-1} *are* P*-matrices.*

Proof Obviously, $\min(A, B) \leq \max(A, B)$. The trick is to use the interval matrix

$$\mathbf{A} = \{\, S \mid \min(A, B) \leq S \leq \max(A, B) \,\}. \tag{1}$$

Then $\mathbf{A}$ is regular by the assumption, and Theorem 1.2 in [6] implies that both $A_1^{-1}A_2$ and $A_1A_2^{-1}$ are P-matrices for each $A_1, A_2 \in \mathbf{A}$. Since both A and B belong to $\mathbf{A}$, the result follows. □

In the next theorem, ϱ denotes the spectral radius.

Theorem 2 *Let* C *be nonsingular,* $D \geq 0$*, and let*

$$0 \leq \alpha < 1/\varrho(|C^{-1}|D). \tag{2}$$

Then all four matrices $(C-\alpha D)^{-1}(C+\alpha D)$*,* $(C-\alpha D)(C+\alpha D)^{-1}$*,* $(C+\alpha D)^{-1}(C-\alpha D)$ *and* $(C+\alpha D)(C-\alpha D)^{-1}$ *are* P*-matrices.*

Proof Put $A = C-\alpha D$, $B = C+\alpha D$, then $A \leq B$, so that $\min(A, B) = A$, $\max(A, B) = B$, and the interval matrix $\mathbf{A}$ defined in (1) becomes

$$\mathbf{A} = \{\, S \mid C - \alpha D \leq S \leq C + \alpha D \,\}.$$

Now, the condition (2), when written in the form

$$\varrho(|C^{-1}|\alpha D) < 1,$$

is precisely Beeck's sufficient condition [1] for regularity of $\mathbf{A}$, and the assertion follows from Theorem 1. □

Theorem 2 can be used in obvious way for generating nonsymmetric P-matrices. Applications of P-matrices in optimization can be found in [5], [4].

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