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Jiří Rohn

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Pod Vodárenskou věží 2, 182 07 Prague 8, phone: +420 266 051 111, fax: +420 286 585 789,
e-mail:rohn@cs.cas.cz

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Abstract:

We give an alternative proof of the Hansen-Bliek optimality result relying on the general theory of interval linear equations.

Keywords:
Interval linear equations, Hansen-Bliek result, alternative proof.

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1 Introduction

Hansen [2] and Bliek [1] published in 1992 almost simultaneously a very nice closed-form expression for the interval hull $[\underline{x}, \overline{x}]$ of the solution set of a system of interval linear equations of the form

$$[I - \Delta, I + \Delta]x = [b_c - \delta, b_c + \delta]$$

(i.e., with $n \times n$ unit midpoint). Both their proofs were not quite rigorous; a rigorous proof was supplied a year later in [6], and the matter was further investigated by Ning and Kearfott [4] and by Neumaier [3].

The fact that the proof given in [6] had been tricky and quite out-of-the-tracks of the established methods of interval analysis intrigued this author for years. Only six years later (the proof below is dated in author's notes as of June 25, 1999), the author found a straightforward proof which delivers the result as a consequence of the general theory described in [5]. The proof is published here with a twelve-years delay, but still in the hope that it will perhaps shed some more light on the matter.

As to the assumption, we note that $\varrho(\Delta) < 1$ is a necessary and sufficient condition for regularity of an interval matrix of the form $[I - \Delta, I + \Delta]$. Y is the set of all ± 1-vectors in $\mathbb{R}^n$, and T_y denotes the diagonal matrix with diagonal vector y.

2 The proof

Theorem 1 *[6] Let $\varrho(\Delta) < 1$. Then for each $i \in \{1, \ldots, n\}$ we have*

$$\underline{x}_i = \min\{\underset{\sim}{x}_i, \nu_i \underset{\sim}{x}_i\},$$

$$\overline{x}_i = \max\{\tilde{x}_i, \nu_i \tilde{x}_i\},$$

where

$$\begin{aligned} \underset{\sim}{x}_i &= -x_i^* + m_{ii}(b_c + |b_c|)_i \\ \tilde{x}_i &= x_i^* + m_{ii}(b_c - |b_c|)_i \\ x_i^* &= (M(|b_c| + \delta))_i \\ \nu_i &= \frac{1}{2m_{ii} - 1} \in (0, 1] \end{aligned}$$

and

$$M = (I - \Delta)^{-1} = (m_{ij}) \geq 0.$$

Proof. Let $i \in \{1, \ldots, n\}$ be fixed. According to the general theory ([5], Theorems 2.2 and 2.4) there holds

$$\overline{x}_i = \max_{y \in Y} (x_y)_i, \tag{2.1}$$

where for each $y \in Y$, x_y is the unique solution of the equation

$$x - T_y \Delta |x| = b_c + T_y \delta. \tag{2.2}$$

We shall prove that the maximum in (2.1) is attained for $y = z$, where z is defined by[2]

$$z_j = \begin{cases} \operatorname{sgn}(b_c)_j & \text{if } j \neq i, \\ 1 & \text{if } j = i \end{cases} \qquad (j = 1, \dots, n). \tag{2.3}$$

To this end, take an arbitrary $y \in Y$. Then, as a solution to (2.2), x_y satisfies

$$x_y = T_y \Delta |x_y| + b_c + T_y \delta, \tag{2.4}$$

hence

$$|x_y|_j \le (\Delta |x_y| + |b_c| + \delta)_j = (\Delta |x_y| + T_z b_c + \delta)_j \tag{2.5}$$

for $j \neq i$, and

$$(x_y)_i \le (\Delta |x_y| + b_c + \delta)_i = (\Delta |x_y| + T_z b_c + \delta)_i, \tag{2.6}$$

which together gives

$$|x_y| + ((x_y)_i - |x_y|_i) e_i \le \Delta |x_y| + T_z b_c + \delta \tag{2.7}$$

and thus also

$$(I - \Delta)|x_y| \le (|x_y|_i - (x_y)_i) e_i + T_z b_c + \delta. \tag{2.8}$$

Premultiplying this inequality by the nonnegative matrix M, we obtain

$$|x_y| \le (|x_y|_i - (x_y)_i) M e_i + M(T_z b_c + \delta) \tag{2.9}$$

and in particular

$$|x_y|_i \le (|x_y|_i - (x_y)_i) m_{ii} + \tilde{x}_i \tag{2.10}$$

since

$$(M(T_z b_c + \delta))_i = (M(|b_c| + ((b_c)_i - |b_c|_i) e_i + \delta))_i = x_i^* + ((b_c)_i - |b_c|_i) m_{ii} = \tilde{x}_i.$$

Now, if $(x_y)_i \ge 0$, then from (2.10) we have $(x_y)_i \le \tilde{x}_i$, and if $(x_y)_i < 0$, then (2.10) yields $(2m_{ii} - 1)(x_y)_i \le \tilde{x}_i$ and thus $(x_y)_i \le \nu_i \tilde{x}_i$ (since $2m_{ii} - 1 \ge 1$ in view of $M = \sum_0^\infty \Delta^j \ge I$), so that

$$(x_y)_i \le \max\{\tilde{x}_i, \nu_i \tilde{x}_i\}. \tag{2.11}$$

On the other hand, if we start in (2.4) with $y = z$, then it follows from the equivalent equation

$$T_z x_z = \Delta |x_z| + T_z b_c + \delta \tag{2.12}$$

and from the definition of z (in particular, $(T_z b_c)_j \ge 0$ and hence $(T_z x_z)_j = |x_z|_j$ for $j \neq i$) that the inequalities (2.5) and (2.6), and thereby also (2.7) through (2.10), hold as equations, so that at the end we obtain

$$|x_z|_i = (|x_z|_i - (x_z)_i) m_{ii} + \tilde{x}_i.$$

Considering separately the cases $(x_z)_i \ge 0$ and $(x_z)_i < 0$ as before, we arrive at

$$(x_z)_i = \max\{\tilde{x}_i, \nu_i \tilde{x}_i\}. \tag{2.13}$$

Hence from (2.1), (2.11) and (2.13) we finally obtain

$$\overline{x}_i = \max_{y \in Y} (x_y)_i = (x_z)_i = \max\{\tilde{x}_i, \nu_i \tilde{x}_i\},$$

which gives the formula for $\overline{x}_i$. The proof for $\underline{x}_i$ is analogous. □

[2] $\operatorname{sgn}(\beta) = 1$ for $\beta \ge 0$, $\operatorname{sgn}(\beta) = -1$ for $\beta < 0$.

On the way, we have also proved the following explicit result which may be useful in some related considerations.

Corollary 2. *Let* $\varrho(\Delta) < 1$. *Then for each* $i \in \{1, \ldots, n\}$ *we have*

$$\overline{x}_i = (x_z)_i,$$

where x_z *is the unique solution of the equation (2.12) and* z *is given by (2.3).*

The result further simplifies under the assumption of nonnegativity of b_c.

Corollary 3. *If* $\varrho(\Delta) < 1$ *and* $b_c \geq 0$, *then*

$$\overline{x} = x_e,$$

where x_e *is the unique solution of the equation*

$$x = \Delta|x| + b_c + \delta.$$

Proof. If $b_c \geq 0$, then $z = e$ independently of i. □

Analogous results hold for $\underline{x}_i$.

In the formulation of Theorem 1 we preferred, as in [6], the use of real arithmetic. Neumaier's result in [3] is formulated in terms of the interval arithmetic.

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