The Solution Set of Interval Linear Equations Is Homeomorphic to the Unit Cube: An Explicit Construction*

J. Rohn†
Institute of Computer Science, Czech Academy of Sciences, Prague, Czech Republic
rohn@cs.cas.cz

November 7, 2015

Abstract

It is proved that the solution set of a system of interval linear equations $\boldsymbol{A}x = \boldsymbol{b}$ with an $n \times n$ regular interval matrix $\boldsymbol{A}$ and a thick right-hand side $\boldsymbol{b}$ is homeomorphic to the unit cube $[-1, 1]^n$, and an explicit homeomorphism is described.

Keywords: interval linear equations, solution set, unit cube, homeomorphism, absolute value equation
AMS subject classifications: 54C05, 65G40

1 Introduction

In general topology, a one-to-one mapping $f : X \to Y$ of a topological space X onto a topological space Y is called *homeomorphism* if both f and f^{-1} are continuous. If there exists a homeomorphism between X and Y, then the two topological spaces are called *homeomorphic*. As the name suggests, homeomorphic topological spaces are considered to be of "similar shape" because they share the same topological properties. Below, when speaking of homeomorphisms of subsets of $\mathbb{R}^n$, we consider these sets to be endowed with topology induced by the standard Euclidean topology of $\mathbb{R}^n$, i.e., open sets in X are just all the sets of the form $X \cap O$, where O is an open set in $\mathbb{R}^n$.

As the main result of this paper we prove that the solution set of a system of interval linear equations is homeomorphic to the unit cube. Given a square interval matrix

$$\boldsymbol{A} = [A_c - \Delta, A_c + \Delta] = \{\, A \mid A_c - \Delta \le A \le A_c + \Delta \,\}$$

which will be assumed throughout to be regular (i.e., each $A \in \boldsymbol{A}$ is nonsingular) and an interval n-vector

$$\boldsymbol{b} = [b_c - \delta, b_c + \delta] = \{\, b \mid b_c - \delta \le b \le b_c + \delta \,\},$$

*Submitted: (insert date); Revised: (insert date); Accepted:(insert date).
†...

the solution set of a formally written system of interval linear equations

$$\boldsymbol{A}x = \boldsymbol{b}$$

is defined as

$$\mathbf{X}(\boldsymbol{A}, \boldsymbol{b}) = \{\, A^{-1}b \mid A \in \boldsymbol{A},\, b \in \boldsymbol{b} \,\}.$$

The basic Oettli-Prager theorem [5] describes the solution set by

$$\mathbf{X}(\boldsymbol{A}, \boldsymbol{b}) = \{\, x \mid |A_c x - b_c| \leq \Delta|x| + \delta \,\}. \tag{1}$$

The presence of an absolute value of x (defined by $|x| = (|x_i|)$ for $x = (x_i)$) on the right-hand side in (1) causes the solution set to be generally of a complicated nonconvex structure (see the example given in Section 4). Yet in Theorem 2.2 below we show that if $\boldsymbol{A}$ is regular $n \times n$ and $\boldsymbol{b}$ is thick (i.e., $\delta > 0$), then there exists an explicit homeomorphism f of $\mathbf{X}(\boldsymbol{A}, \boldsymbol{b})$ onto the unit cube $[-1, 1]^n$. We have not been able to express the inverse homeomorphism f^{-1} by a closed-form formula, but we present an algorithm which for each $y \in [-1, 1]^n$ computes the value of $f^{-1}(y)$ in a finite number of steps. A 2×2 example is given in Section 4 to demonstrate the workings of f and f^{-1}. Finally in Section 5 we show by means of a counterexample that the assumption of $\delta > 0$ in Theorem 2.2 cannot be dropped. We use the notation

$$T_y = \begin{pmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_n \end{pmatrix};$$

I is the identity matrix and e is the vector of all ones. Notice that $[-1, 1]^n = [-e, e]$.

2 The homeomorphism

In the proof of the main theorem we shall essentially use the following result proved in [7, Prop. 4.2].

Theorem 2.1. *Let A, $B \in \mathbb{R}^{n\times n}$ and let the interval matrix $[A - |B|,\, A + |B|]$ be regular. Then for each right-hand side $b \in \mathbb{R}^n$ the absolute value equation*

$$Ax + B|x| = b \tag{2}$$

has a unique solution.

Now the main result of this paper is formulated as follows.

Theorem 2.2. *Let $\boldsymbol{A}$ be regular and let $\delta > 0$. Then the mapping*

$$f(x) = \left(\frac{(A_c x - b_c)_i}{(\Delta|x| + \delta)_i} \right)_{i=1}^{n} \tag{3}$$

is a homeomorphism of $\mathbf{X}(\boldsymbol{A}, \boldsymbol{b})$ onto the unit cube $[-1, 1]^n$.

Proof: We shall carry out the proof in several steps. Let $Y = [-1,1]^n$.

(a) f *maps* $\mathbf{X}(\boldsymbol{A},\boldsymbol{b})$ *into* Y. It follows from the Oettli-Prager description (1) that

$$-1 \le \frac{(A_c x - b_c)_i}{(\Delta|x|+\delta)_i} \le 1 \qquad (i = 1,\dots,n)$$

holds for each $x \in \mathbf{X}(\boldsymbol{A},\boldsymbol{b})$, hence f maps $\mathbf{X}(\boldsymbol{A},\boldsymbol{b})$ into Y.

(b) f *maps* $\mathbf{X}(\boldsymbol{A},\boldsymbol{b})$ *onto* Y. Let $y \in Y$. Then $|T_y\Delta| \le \Delta$, hence

$$[A_c - |T_y\Delta|, A_c + |T_y\Delta|] \subseteq [A_c - \Delta, A_c + \Delta],$$

so that regularity of $[A_c - \Delta, A_c + \Delta]$ implies that of $[A_c - |T_y\Delta|, A_c + |T_y\Delta|]$, hence by Theorem 2.1 the nonlinear equation

$$A_c x - T_y\Delta|x| = b_c + T_y\delta \tag{4}$$

has a unique solution x. This x belongs to $\mathbf{X}(\boldsymbol{A},\boldsymbol{b})$ because

$$|A_c x - b_c| = |T_y(\Delta|x| + \delta)| \le \Delta|x| + \delta,$$

and

$$\frac{(A_c x - b_c)_i}{(\Delta|x|+\delta)_i} = y_i$$

holds for each i by (4), hence $f(x) = y$.

(c) f *is one-to-one.* If $f(x') = f(x'') = y$ for some $x', x'' \in \mathbf{X}(\boldsymbol{A},\boldsymbol{b})$, then both x' and x'' must solve the equation (4), so that from the above-mentioned uniqueness of its solution it follows that $x' = x''$, which proves that f is bijective.

(d) f *and* f^{-1} *are continuous.* Continuity of f follows from (3). We prove continuity of f^{-1} by contradiction. Assume that f^{-1} is not continuous at some $y \in Y$, so that there exists a sequence $\{y_j\}$ of points of Y such that $y_j \to y$, but $f^{-1}(y_j) \not\to f^{-1}(y)$. Denote $x = f^{-1}(y)$ and $x_j = f^{-1}(y_j)$ for each j. Since $x_j \not\to x$, by definition of limit there exists an $\varepsilon > 0$ and a subsequence $\{x_{j_k}\}$ such that

$$\|x_{j_k} - x\|_2 \ge \varepsilon \tag{5}$$

for each k (we use the Euclidean norm $\|x\|_2 = \sqrt{x^T x}$). Because $\mathbf{X}(\boldsymbol{A},\boldsymbol{b})$ is compact (Beeck [1]), $\{x_{j_k}\}$ contains a convergent subsequence $\{x_{j_{k_\ell}}\}$, $x_{j_{k_\ell}} \to x^* \in \mathbf{X}(\boldsymbol{A},\boldsymbol{b})$, and taking the limit in (5) for $j_{k_\ell} \to \infty$ yields $\|x^* - x\|_2 \ge \varepsilon > 0$, so that $x^* \ne x$. Now, $x_{j_{k_\ell}} \to x^*$ in view of continuity of f implies $y_{j_{k_\ell}} = f(x_{j_{k_\ell}}) \to f(x^*)$, but also $y_{j_{k_\ell}} \to y$, hence $f(x^*) = y = f(x)$ where $x^* \ne x$, a contradiction.

This proves that f is a homeomorphism of $\mathbf{X}(\boldsymbol{A},\boldsymbol{b})$ onto Y. $\square$

We note that part (d) of the proof follows from a more general assertion stating that if f is a continuous one-to-one mapping of a compact set $X \subset \mathbb{R}^n$ onto $Y \subset \mathbb{R}^n$, then Y is also compact and f^{-1} is continuous, so that f is a homeomorphism (see [9, Thms. 4.13, 4.17]). But we have preferred to deliver an elementary proof. In part (b) of the proof we have simultaneously proved the following characterization of f^{-1}.

Theorem 2.3. *Under assumptions and notation of Theorem 2.2, for each $y \in [-1,1]^n$ we have*

$$f^{-1}(y) = x,$$

where x is the unique solution of the equation

$$A_c x - T_y\Delta|x| = b_c + T_y\delta. \tag{6}$$

As a simple consequence of the main result we obtain that under our assumptions any two solution sets of interval linear equations are homeomorphic.

Theorem 2.4. *If $\boldsymbol{A}$, $\boldsymbol{A}'$ are regular interval matrices of the same size and $\boldsymbol{b}$, $\boldsymbol{b}'$ are matching thick interval vectors, then the solution sets $\mathbf{X}(\boldsymbol{A},\boldsymbol{b})$ and $\mathbf{X}(\boldsymbol{A}',\boldsymbol{b}')$ are homeomorphic.*

Proof: If $f : \mathbf{X}(\boldsymbol{A},\boldsymbol{b}) \to [-1,1]^n$ and $g : \mathbf{X}(\boldsymbol{A}',\boldsymbol{b}') \to [-1,1]^n$ are homeomorphisms whose existence under our assumptions is guaranteed by Theorem 2.2, then $g^{-1} \circ f$ is a homeomorphism between $\mathbf{X}(\boldsymbol{A},\boldsymbol{b})$ and $\mathbf{X}(\boldsymbol{A}',\boldsymbol{b}')$. $\square$

Notice that the unit cube itself is a solution set of interval linear equations. Indeed, it is the solution set of

$$[I, I]x = [-e, e].$$

If we introduce the Hadamard division of two vectors $a = (a_i)$, $b = (b_i) \in \mathbb{R}^n$ by

$$\frac{a}{b} = \left(\frac{a_i}{b_i}\right)_{i=1}^{n}$$

(assuming that $b_i \neq 0$ for each i), then we may rewrite (3) in a "vectorized" form

$$f(x) = \frac{A_c x - b_c}{\Delta|x| + \delta}.$$

3 The inverse homeomorphism

By Theorem 2.3, for each $y \in [-1,1]^n$ there holds $f^{-1}(y) = x$, where x is the unique solution of the equation (6). This equation can be efficiently solved by the algorithm `absvaleqn` described in [7], [8] which is attached here in the form of an executable MATLAB file (see next page). The algorithm is invoked by

```
[x,S]=absvaleqn(A,B,b)
```

and in a finite number of steps it returns either a solution x of (2), or a singular matrix S satisfying $|S - A| \leq |B|$. In our case regularity of $\boldsymbol{A}$ precludes existence of such a singular matrix, hence we get x simply by using

```
x=absvaleqn(Ac,-diag(y)*Delta,bc+diag(y)*delta)
```

where `Ac, Delta, bc` and `delta` are MATLAB variables corresponding to A_c, Δ, b_c and δ, respectively. As regards the number of steps (passes through the `while ... end` loop of the algorithm), it has been shown in [7] on a set of randomly generated 100,000 examples of various sizes that the average number of steps was about $0.1n$, where n is the matrix size.

In [6] we denoted the solution of (6) for $y \in \{-1,1\}^n$ by x_y and we proved there that the interval hull $\boldsymbol{x}(\boldsymbol{A},\boldsymbol{b})$ of the solution set $\mathbf{X}(\boldsymbol{A},\boldsymbol{b})$ (i.e., the minimal enclosure of $\mathbf{X}(\boldsymbol{A},\boldsymbol{b})$ w.r.t. inclusion) is equal to the interval hull of the finite set of x_y's. In this way we may reformulate the result as

$$\boldsymbol{x}(\boldsymbol{A},\boldsymbol{b}) = [\min_{y\in\{-1,1\}^n} f^{-1}(y), \max_{y\in\{-1,1\}^n} f^{-1}(y)] \tag{7}$$

(entrywise minimum/maximum); notice that $\{-1,1\}^n$, not $[-1,1]^n$, is used in (7).

Observe also that $f^{-1}(0) = A_c^{-1} b_c$.

```
function [x,S,iter]=absvaleqn(A,B,b) % ABSolute VALue EQuatioN
%
% ~isempty(x): x solves A*x+B*abs(x)=b (A,B square), S is empty,
% ~isempty(S): S is a singular matrix satisfying abs(S-A)<=abs(B), x is empty.
%  iter: number of iterations (it may be zero).
%
%  Copyright 2005-2013 Jiri Rohn (rohn@cs.cas.cz)
%
b=b(:); n=length(b); I=eye(n,n);
ep=n*(max([norm(A,inf) norm(B,inf) norm(b,inf)]))*eps;
x=[]; S=[]; iter=0;
if rank(A)<n, S=A; return, end
x=A\b; z=sgn(x);
if rank(A+B*diag(z))<n, S=A+B*diag(z); x=[]; return, end
x=(A+B*diag(z))\b;
C=-inv(A+B*diag(z))*B;
X=zeros(n,n);
r=zeros(1,n);
while any(z.*x<-ep)
    k=find(z.*x<-ep,1);
    iter=iter+1;
    if 1+2*z(k)*C(k,k)<=0
        S=A+B*(diag(z)+(1/C(k,k))*I(:,k)*I(k,:)); x=[]; return
    end
    if ((k<n)&&(all(r(k)>r(k+1:n))))||((k==n)&&(r(k)>0))
        x=x-X(:,k); z=sgn(x);
        ct=A*x;
        jm=abs(B)*abs(x);
        y=zeros(1,n);
        for i=1:n
            if jm(i)>ep, y(i)=ct(i)/jm(i); else y(i)=1; end
        end
        S=A-diag(y)*abs(B)*diag(z); x=[]; return
    end
    X(:,k)=x;
    r(k)=iter;
    z(k)=-z(k);
    alpha=2*z(k)/(1-2*z(k)*C(k,k));
    x=x+alpha*x(k)*C(:,k);
    C=C+alpha*C(:,k)*C(k,:);
end
function z=sgn(x) % SiGN vector of x (column)
n=length(x);
z=zeros(n,1);
for j=1:n
    if x(j)>=0, z(j)=1; else z(j)=-1; end
end
```

4 Example

To visualize workings of both homeomorphisms, consider the well-known Hansen's example in [3]

$$\begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} x = \begin{pmatrix} [\ 0,120] \\ [60,240] \end{pmatrix}.$$

The left-hand side interval matrix is regular by Beeck's criterion [2] because $\varrho(|A_c^{-1}|\Delta) = 0.6364 < 1$, and $\delta > 0$.

First, we performed the following procedure (only relevant part of the code is shown):

```
for i=1:100000
    A=Ac+(2*rand(n,n)-1).*Delta;
    b=bc+(2*rand(n,1)-1).*delta;
    x=A\b;
    y=(Ac*x-bc)./(Delta*abs(x)+delta);
    plot(y(1),y(2));
end
```

($n = 2$ throughout). As it can be seen, the file constructs 100,000 random solutions $x \in \mathbf{X}(\boldsymbol{A}, \boldsymbol{b})$, and for each of them it evaluates $y = f(x)$ and plots y. Figure 1 depicts the plot which vaguely resembles a fingerprint. It demonstrates that all the points plotted belong to $[-1,1]^2$, the unit "cube" (rather, square) in $\mathbb{R}^2$.

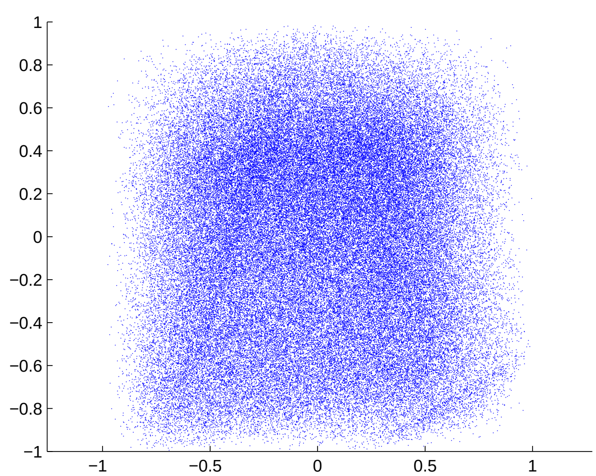

Figure 1: Plot of $f(X')$ for a random $X' \subset \mathbf{X}(\boldsymbol{A}, \boldsymbol{b})$ with card(X')=100,000.

Second, we used the following procedure to construct 100,000 random points in $[-1,1]^2$, and for each of them to compute $x = f^{-1}(y)$ using the above-described file `absvaleqn` and to plot x:

```
for i=1:100000
    y=2*rand(n,1)-1;
    x=absvaleqn(Ac,-diag(y)*Delta,bc+diag(y)*delta);
    plot(x(1),x(2));
end
```

The result is shown in Figure 2. The graph is remarkably sharp and dense, in particular in the center of it. Statistical modelling of solution sets was also performed, in another context, by Shary [12].

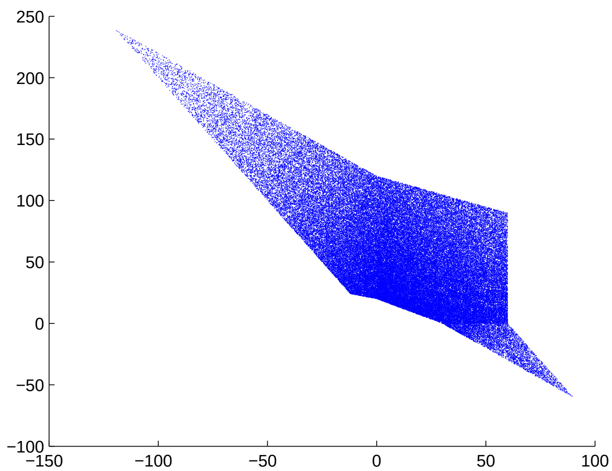

Figure 2: Plot of $f^{-1}(Y')$ for a random $Y' \subset [-1,1]^2$ with card(Y')=100,000.

And third, we plotted in Figure 3 the solution set $\mathbf{X}(\boldsymbol{A},\boldsymbol{b})$ using very nice specialized software by Irene A. Sharaya [10] whose theoretical basis is described in [11]. It shows that the graph in Figure 2 truly reflects, up to the scale, the actual shape of the solution set $\mathbf{X}(\boldsymbol{A},\boldsymbol{b})$. The eight points and emphasized intersections with coordinate axes were added by the software automatically.

5 Nonexistence of a homeomorphism

Finally we shall show that the assumption of $\delta > 0$ in Theorem 2.2 cannot be dropped. To this end, consider the following example from Neumaier's book [4] (a "butterfly").

$$\begin{pmatrix} [\ \ 2,4] & [-1,1] \\ [-1,1] & [\ \ 2,4] \end{pmatrix} x = \begin{pmatrix} [-3,3] \\ [0,0] \end{pmatrix} \tag{8}$$

Theorem 5.1. *There exists no homeomorphism between the solution set* $\mathbf{X}(\boldsymbol{A},\boldsymbol{b})$ *of (8) and the unit cube* $[-1,1]^2$.

Proof: Assume to the contrary that such a homeomorphism $g : \mathbf{X}(\boldsymbol{A},\boldsymbol{b}) \to [-1,1]^2$ exists. From Fig. 4 we can see that the three points $(-1,0)$, $(0,0)$ and $(1,0)$ belong

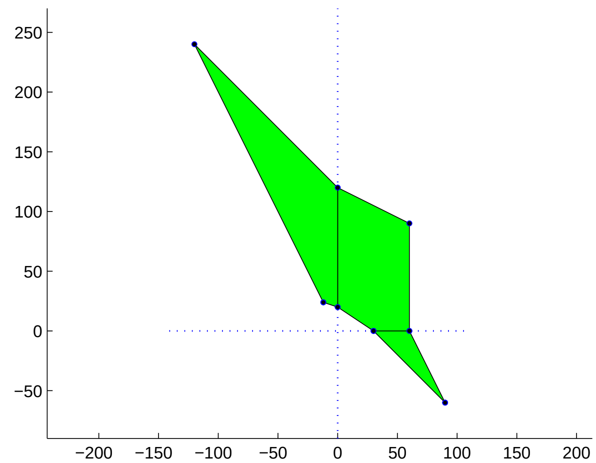

Figure 3: Plot of $\mathbf{X}(\boldsymbol{A}, \boldsymbol{b})$ using specialized software.

to $\mathbf{X}(\boldsymbol{A}, \boldsymbol{b})$. Let $y^1 = g(-1, 0)$ and $y^2 = g(1, 0)$, and choose a point $y^3 \in [-1, 1]^2$ such that $g(0, 0)$ belongs neither to the segment connecting y^1 with y^3, nor to the segment connecting y^3 with y^2. Define a continuous mapping $\varphi : [0, 1] \to [-1, 1]^2$ by

$$\varphi(t) = \begin{cases} y^1 + 2t(y^3 - y^1) & \text{if } t \in [0, \frac{1}{2}], \\ y^3 + (2t - 1)(y^2 - y^3) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} .$$

Then by convexity of $[-1, 1]^2$ we have $\varphi(t) \in [-1, 1]^2$ for each $t \in [0, 1]$ and by the choice of y^3 we have that $\varphi(t) \neq g(0, 0)$ for each $t \in [0, 1]$. Now the real-valued function of one real variable

$$\psi(t) = (g^{-1}(\varphi(t)))_1, \qquad t \in [0, 1]$$

satisfies

$$\psi(0) = (g^{-1}(y^1))_1 = (-1, 0)_1 = -1$$

and

$$\psi(1) = (g^{-1}(y^2))_1 = (1, 0)_1 = 1,$$

hence by the intermediate value theorem there exists a $t^* \in [0, 1]$ such that

$$\psi(t^*) = (g^{-1}(\varphi(t^*)))_1 = 0.$$

Hence the first coordinate of the point $g^{-1}(\varphi(t^*)) \in \mathbf{X}(\boldsymbol{A}, \boldsymbol{b})$ is zero, and from Fig. 4 we can see that there is exactly one such point, namely $(0, 0) \in \mathbf{X}(\boldsymbol{A}, \boldsymbol{b})$. Thus $g^{-1}(\varphi(t^*)) = (0, 0)$ and consequently

$$\varphi(t^*) = g(0, 0)$$

which runs contrary to the definition of the mapping φ which was constructed so as to $\varphi(t) \neq g(0, 0)$ for each $t \in [0, 1]$. This contradiction shows that a homeomorphism $g : \mathbf{X}(\boldsymbol{A}, \boldsymbol{b}) \to [-1, 1]^2$ does not exist. $\square$

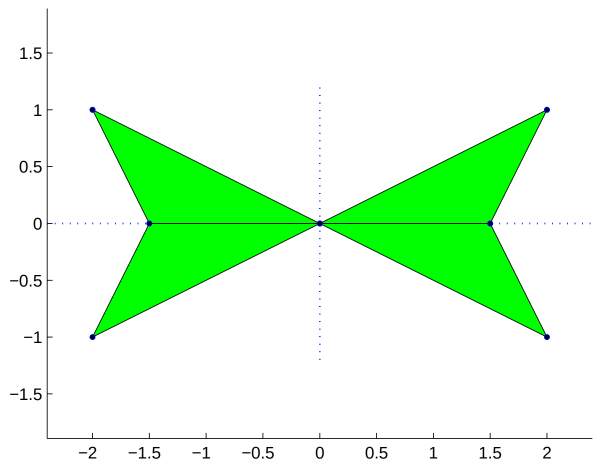

Figure 4: Plot of $\mathbf{X}(\boldsymbol{A}, \boldsymbol{b})$ for (8).

6 Conclusion

We have constructed an explicit homeomorphism of the solution set (sometimes called the "united solution set") onto the unit cube. There exist also other types of solution sets, as e.g. the tolerance solution set, the control solution set, or Shary's AE-solution sets. The question whether some of them possess the aforementioned property remains open.

Acknowledgment

The author wishes to thank Prof. S. P. Shary, Prof. A. V. Lakeyev and two anonymous referees for helpful suggestions that resulted in essential improvement of the text of the paper.

References

[1] H. Beeck. Charakterisierung der Lösungsmenge von Intervallgleichungssystemen. *Zeitschrift für Angewandte Mathematik und Mechanik*, 53:T181–T182, 1973.

[2] H. Beeck. Zur Problematik der Hüllenbestimmung von Intervallgleichungssystemen. In K. Nickel, editor, *Interval Mathematics*, Lecture Notes in Computer Science 29, pages 150–159, Berlin, 1975. Springer-Verlag.

[3] E. Hansen. On linear algebraic equations with interval coefficients. In E. Hansen, editor, *Topics in Interval Analysis*, pages 33–46, Oxford, 1969. Oxford University Press.

[4] A. Neumaier. *Interval Methods for Systems of Equations*. Cambridge University Press, Cambridge, 1990.

[5] W. Oettli and W. Prager. Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides. *Numerische Mathematik*, 6:405–409, 1964.

[6] J. Rohn. Systems of linear interval equations. *Linear Algebra and Its Applications*, 126:39–78, 1989.

[7] J. Rohn. An algorithm for solving the absolute value equation. *Electronic Journal of Linear Algebra*, 18:589–599, 2009. `http://www.math.technion.ac.il/iic/ela/ela-articles/articles/vol18_pp589-599.pdf`.

[8] J. Rohn. An algorithm for solving the absolute value equation: An improvement. Technical Report 1063, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, January 2010. `http://uivtx.cs.cas.cz/~rohn/publist/absvaleqnreport.pdf`.

[9] W. Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, New York, 1976.

[10] I. A. Sharaya. Intlininc2d – a software package for visualization of the solution sets to interval linear systems of relations with two unknowns. Version for MATLAB: Release 01.09.2014. Available at http://www.nsc.ru/interval/sharaya/index.html.

[11] I. A. Sharaya. Boundary intervals method for visualization of polyhedral solution sets. *Reliable Computing*, 19:435–467, 2015.

[12] S. P. Shary. Interval analysis or Monte-Carlo methods? (in Russian). *Vychislitel'nye Technologii*, 12:103–115, 2007.