

EXPLICIT INVERSE OF AN INTERVAL MATRIX WITH UNIT MIDPOINT*

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Abstract. Explicit formulae for the inverse of an interval matrix of the form $[I - \Delta, I + \Delta]$ (where I is the unit matrix) are proved via finding explicit solutions of certain nonlinear matrix equations.

Key words. Interval matrix, Unit midpoint, Inverse interval matrix, Regularity.

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1. Introduction. In this paper, we study the inverse of an interval matrix of a special form

$$\mathbf{A} = [I - \Delta, I + \Delta] \quad (1.1)$$

(i.e., having the unit midpoint). Computing the inverse interval matrix (defined in Section 3) is NP-hard in general (Coxson [1]). Yet it was shown in [9, Theorem 2] that in the special case of an interval matrix of the form (1.1), the inverse interval matrix can be expressed by simple formulae in terms of the matrix

$$M = (I - \Delta)^{-1}$$

(see Theorem 4.4 below). The result was proved there as an application of a very special assertion on interval linear equations. In this paper, we give another proof of this theorem making use of a general result (Theorem 3.5) according to which the inverse of an $n \times n$ interval matrix can be computed from unique solutions of 2^n nonlinear matrix equations. As the main result of this paper, we show in Theorem 4.2 that for interval matrices of the form (1.1), the unique solution of each of these 2^n nonlinear equations can be expressed explicitly; this, in turn, makes it possible to express the inverse of (1.1) explicitly, as showed in the proof of Theorem 4.4. Moreover, this approach also allows us to specify the matrices in \mathbf{A} at whose inverses the componentwise bounds on \mathbf{A}^{-1} are attained (Theorem 5.1).

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The paper is organized as follows. In Section 2, we sum up the notation used. In Section 3, the inverse interval matrix is defined and a general (finite, but exponential) method for its computation is given. The explicit solutions of the respective nonlinear equations are described in Section 4 and are then used for deriving explicit formulae for the inverse of (1.1). Finally, in Section 5, matrices are described at whose inverses the componentwise bounds on the interval inverse are attained.

2. Notation. We use the following notation. A_{ij} denotes the ij th entry and $A_{\bullet j}$ the j th column of A . Matrix inequalities, as $A \leq B$ or $A < B$, are understood componentwise. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. The same notation also applies to vectors that are considered one-column matrices. I is the unit matrix, e_j denotes its j th column, and $e = (1, \dots, 1)^T$ is the vector of all ones. $Y = \{y \mid |y| = e\}$ is the set of all ± 1 -vectors in \mathbb{R}^n , so that its cardinality is 2^n . For each $y \in \mathbb{R}^n$, we denote

$$T_y = \text{diag}(y_1, \dots, y_n) = \begin{pmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_n \end{pmatrix}.$$

Finally, we introduce the *real* spectral radius of a square matrix A by

$$\varrho_0(A) = \max\{|\lambda| \mid \lambda \text{ is a real eigenvalue of } A\}, \quad (2.1)$$

and we set $\varrho_0(A) = 0$ if no real eigenvalue exists; $\varrho(A)$ is the usual spectral radius of A .

3. Inverse interval matrix. Given two $n \times n$ matrices A_c and Δ , $\Delta \geq 0$, the set of matrices

$$\mathbf{A} = \{A \mid |A - A_c| \leq \Delta\}$$

is called a (*square*) *interval matrix* with midpoint matrix A_c and radius matrix Δ . Since the inequality $|A - A_c| \leq \Delta$ is equivalent to $A_c - \Delta \leq A \leq A_c + \Delta$, we can also write

$$\mathbf{A} = \{A \mid \underline{A} \leq A \leq \overline{A}\} = [\underline{A}, \overline{A}],$$

where $\underline{A} = A_c - \Delta$ and $\overline{A} = A_c + \Delta$ are called the *bounds* of \mathbf{A} .

DEFINITION 3.1. A square interval matrix \mathbf{A} is called *regular* if each $A \in \mathbf{A}$ is nonsingular, and it is said to be *singular* otherwise (i.e., if it contains a singular matrix).

Many necessary and sufficient regularity conditions are known (the paper [11] surveys forty of them). We shall use here the following one (condition (xxxiv) in [11]; see (2.1) for the definition of ϱ_0).

PROPOSITION 3.2. *A square interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ is regular if and only if A_c is nonsingular and*

$$\max_{y,z \in Y} \varrho_0(A_c^{-1} T_y \Delta T_z) < 1.$$

DEFINITION 3.3. For a regular interval matrix \mathbf{A} , we define its *inverse interval matrix* $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ by

$$\begin{aligned} \underline{B} &= \min \{ A^{-1} \mid A \in \mathbf{A} \}, \\ \overline{B} &= \max \{ A^{-1} \mid A \in \mathbf{A} \} \end{aligned}$$

(componentwise).

COMMENT 3.4. The above definition means that for every $i, j = 1, \dots, n$,

$$\underline{B}_{ij} = \min \{ (A^{-1})_{ij} \mid A \in \mathbf{A} \}, \quad (3.1)$$

$$\overline{B}_{ij} = \max \{ (A^{-1})_{ij} \mid A \in \mathbf{A} \}. \quad (3.2)$$

Since \mathbf{A} is regular, the mapping $A \mapsto A^{-1}$ is continuous in \mathbf{A} and all the minima and maxima in (3.1), (3.2) are attained. Thus, \mathbf{A}^{-1} is the narrowest interval matrix enclosing the set of matrices $\{A^{-1} \mid A \in \mathbf{A}\}$. For more results on the inverse interval matrix, see Hansen [2], Hansen and Smith [3], Herzberger and Bethke [4], and Rohn [8, 10]. Computing the inverse interval matrix is NP-hard (Coxson [1]).

We have the following general result (see Theorem 5.1, Assertion (A3) and Theorem 6.2 of [8]).

THEOREM 3.5. *Let $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ be regular. Then for each $y \in Y$, the matrix equation*

$$A_c B - T_y \Delta |B| = I$$

has a unique matrix solution B_y , and for the inverse interval matrix $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ we have (componentwise)

$$\begin{aligned} \underline{B} &= \min \{ B_y \mid y \in Y \}, \\ \overline{B} &= \max \{ B_y \mid y \in Y \}. \end{aligned}$$

Thus, in contrast to the definition, only a finite number of matrices B_y , $y \in Y$ (albeit 2^n of them) are needed to compute the inverse interval matrix. In the next section, we shall show that in the case of $A_c = I$, all the matrices B_y can be expressed explicitly.

4. Inverse interval matrix with unit midpoint. From now on, we shall consider interval matrices of the form

$$\mathbf{A} = [I - \Delta, I + \Delta], \quad (4.1)$$

i.e., with unit midpoint I . First, we shall resolve the question of regularity of (4.1).

PROPOSITION 4.1. *An interval matrix (4.1) is regular if and only if*

$$\varrho(\Delta) < 1. \quad (4.2)$$

Proof. For each $y, z \in Y$, we have

$$\varrho_0(T_y \Delta T_z) \leq \varrho(T_y \Delta T_z) \leq \varrho(|T_y \Delta T_z|) = \varrho(\Delta) = \varrho_0(\Delta) = \varrho_0(T_e \Delta T_e)$$

(the equation $\varrho(\Delta) = \varrho_0(\Delta)$ is a consequence of the Perron-Frobenius theorem [5]). Hence,

$$\max_{y, z \in Y} \varrho_0(T_y \Delta T_z) = \varrho(\Delta),$$

and the assertion follows from Proposition 3.2. \square

As is well known, the condition $\varrho(\Delta) < 1$ implies

$$(I - \Delta)^{-1} = \sum_{j=0}^{\infty} \Delta^j \geq I \geq 0$$

(because Δ is nonnegative). Set

$$M = (I - \Delta)^{-1} = (m_{ij})$$

and

$$\mu = (\mu_j),$$

where

$$\mu_j = \frac{m_{jj}}{2m_{jj} - 1} \quad (j = 1, \dots, n). \quad (4.3)$$

Then $M \geq 0$, and consequently, we have

$$m_{ij} \geq 0, \quad (4.4)$$

$$m_{jj} \geq 1, \quad (4.5)$$

$$2m_{jj} - 1 \geq 1, \quad (4.6)$$

$$2\mu_j - 1 \in (0, 1], \quad (4.7)$$

$$\mu_j \in (\frac{1}{2}, 1], \quad (4.8)$$

$$\mu_j \leq m_{jj}, \quad (4.9)$$

$$(2\mu_j - 1)m_{jj} = \mu_j \quad (4.10)$$

for every $i, j = 1, \dots, n$, and also

$$M\Delta = \Delta M = \sum_{j=1}^{\infty} \Delta^j = M - I. \quad (4.11)$$

These simple facts will be utilized in the proofs to follow.

Under the assumption (4.2), the interval matrix (4.1) is regular. Hence, by Theorem 3.5, the equation

$$B - T_y \Delta |B| = I$$

has a unique solution B_y for each $y \in Y$. We shall now show that this B_y can be expressed explicitly.

THEOREM 4.2. *Let $\varrho(\Delta) < 1$. Then for each $y \in Y$, the unique solution of the matrix equation*

$$B - T_y \Delta |B| = I \quad (4.12)$$

is given by

$$B_y = T_y M T_y + T_y (M - I) T_y (I - T_y), \quad (4.13)$$

i.e., componentwise, for every $i, j = 1, \dots, n$,

$$(B_y)_{ij} = y_i y_j m_{ij} + y_i (1 - y_j) (m_{ij} - I_{ij}) \mu_j, \quad (4.14)$$

or equivalently,

$$(B_y)_{ij} = \begin{cases} y_i m_{ij} & \text{if } y_j = 1, \\ y_i (2\mu_j - 1) m_{ij} & \text{if } y_j = -1 \text{ and } i \neq j, \\ \mu_j & \text{if } y_j = -1 \text{ and } i = j, \end{cases} \quad (4.15)$$

or equivalently,

$$(B_y)_{ij} = \frac{(y_i + (1 - y_i) I_{ij}) m_{ij}}{y_j + (1 - y_j) m_{jj}}. \quad (4.16)$$

COMMENT 4.3. We give two proofs of this theorem. The first one shows how the formulae (4.13)–(4.16) can be derived. The second one demonstrates that once they are known, it is relatively simple to verify that B_y given by them is indeed a solution to (4.12). As it can be expected, the first proof is essentially longer, but more informative.

First Proof of Theorem 4.2. Under the assumption (4.2), it follows from Theorem 3.5 that the equation (4.12) has a unique solution B_y . Fix a $j \in \{1, \dots, n\}$ and set

$$x = T_y(B_y)_{\bullet j} \quad (4.17)$$

(where $(B_y)_{\bullet j}$ is the j th column of B_y). Then from (4.12), if written in the form

$$T_y B - \Delta |T_y B| = T_y$$

(because $|T_y B| = |B|$), it follows that x satisfies the equation

$$x - \Delta |x| = y_j e_j. \quad (4.18)$$

If $y_j = 1$, then $x = \Delta |x| + e_j \geq 0$. Hence, $|x| = x$ and from (4.18) we have simply $x = (I - \Delta)^{-1} e_j = M e_j$. Thus,

$$x_i = m_{ij} \quad (4.19)$$

for each i . Now, let $y_j = -1$. Then from (4.18), it follows that $x_i \geq 0$ for each $i \neq j$, so that we can write

$$|x| = (x_1, \dots, x_{j-1}, |x_j|, x_{j+1}, \dots, x_n)^T = x + (|x_j| - x_j) e_j,$$

and from (4.18), we obtain

$$(I - \Delta)x = -e_j + (|x_j| - x_j) \Delta e_j.$$

Hence, premultiplying this equation by the nonnegative matrix $M = (I - \Delta)^{-1}$ gives

$$x = -M e_j + (|x_j| - x_j) M \Delta e_j = -M e_j + (|x_j| - x_j) (M - I) e_j \quad (4.20)$$

(using (4.11)), and consequently,

$$x_j = -m_{jj} + (|x_j| - x_j)(m_{jj} - 1). \quad (4.21)$$

Assuming $x_j \geq 0$, we would have $x_j = -m_{jj} \leq -1 < 0$ by (4.5), a contradiction. This shows that $x_j < 0$. Hence, $|x_j| = -x_j$ and (4.21) yields

$$x_j = -\frac{m_{jj}}{2m_{jj} - 1} = -\mu_j \quad (4.22)$$

(see (4.3)). Thus,

$$|x_j| - x_j = -2x_j = 2\mu_j,$$

and substituting into (4.20) gives

$$x = -M e_j + 2\mu_j (M - I) e_j,$$

so that

$$x_i = -m_{ij} + 2\mu_j m_{ij} = (2\mu_j - 1)m_{ij} \quad (4.23)$$

for each $i \neq j$. Hence, from (4.19), (4.23) and (4.22), we obtain that

$$x_i = \begin{cases} m_{ij} & \text{if } y_j = 1, \\ (2\mu_j - 1)m_{ij} & \text{if } y_j = -1 \text{ and } i \neq j, \\ -\mu_j & \text{if } y_j = -1 \text{ and } i = j \end{cases}$$

for each i . Since $(B_y)_{\bullet j} = T_y x$ by (4.17), this means that

$$(B_y)_{ij} = y_i x_i = \begin{cases} y_i m_{ij} & \text{if } y_j = 1, \\ y_i (2\mu_j - 1)m_{ij} & \text{if } y_j = -1 \text{ and } i \neq j, \\ \mu_j & \text{if } y_j = -1 \text{ and } i = j, \end{cases}$$

which is (4.15). Hence, we can see that $(B_y)_{ij}$, aside from m_{ij} and μ_j , depends on y_i and y_j only. The values of $(B_y)_{ij}$ for all possible combinations of y_i and y_j are summed up in Table 4.1.

y_i	y_j	$(B_y)_{ij}$
1	1	m_{ij}
-1	1	$-m_{ij}$
1	-1	$(2\mu_j - 1)m_{ij}$
-1	-1	$-(2\mu_j - 1)m_{ij} + 2\mu_j I_{ij}$

TABLE 4.1

Validity of (4.14), (4.16) can be checked simply by assigning $y_i = \pm 1$, $y_j = \pm 1$ into their right-hand sides and verifying that the results obtained correspond to those in Table 4.1. Finally, rewriting (4.14) in the equivalent form

$$(B_y)_{ij} = y_i m_{ij} y_j + y_i (m_{ij} - I_{ij}) \mu_j (1 - y_j),$$

we can see that this is the componentwise version of (4.13) (taking into account that all three matrices T_y , T_μ , I are diagonal). \square

Second Proof of Theorem 4.2. Equivalence of (4.13), (4.14), (4.15) and (4.16) has been established in the previous proof. From (4.15), (4.7) and (4.10), we have

$$|B_y|_{ij} = \begin{cases} m_{ij} & \text{if } y_j = 1, \\ (2\mu_j - 1)m_{ij} & \text{if } y_j = -1 \end{cases}$$

for each i, j , but also

$$(M(T_y + T_\mu(I - T_y)))_{ij} = m_{ij}(y_j + \mu_j(1 - y_j)) = \begin{cases} m_{ij} & \text{if } y_j = 1, \\ (2\mu_j - 1)m_{ij} & \text{if } y_j = -1 \end{cases}$$

for each i, j , which shows that

$$|B_y| = M(T_y + T_\mu(I - T_y)). \quad (4.24)$$

Then

$$\begin{aligned} B_y - I &= T_y(M - I)T_y + T_y(M - I)T_\mu(I - T_y) = T_y(M - I)(T_y + T_\mu(I - T_y)) \\ &= T_y\Delta M(T_y + T_\mu(I - T_y)) = T_y\Delta|B_y| \end{aligned}$$

(because of (4.11) and of the fact that $T_y^2 = I$), and hence

$$B_y - T_y\Delta|B_y| = I.$$

This means that B_y is a solution to (4.12) which, according to Theorem 3.5, is unique. \square

Now we shall apply this result to the inverse interval matrix.

THEOREM 4.4. *Let $\mathbf{A} = [I - \Delta, I + \Delta]$ with $\rho(\Delta) < 1$. Then the inverse interval matrix $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ is given by*

$$\underline{B} = -M + T_\kappa, \quad (4.25)$$

$$\overline{B} = M, \quad (4.26)$$

where

$$\kappa_j = \frac{2m_{jj}^2}{2m_{jj} - 1} \quad (j = 1, \dots, n), \quad (4.27)$$

or componentwise

$$\underline{B}_{ij} = \begin{cases} -m_{ij} & \text{if } i \neq j, \\ \mu_j & \text{if } i = j, \end{cases} \quad (4.28)$$

$$\overline{B}_{ij} = m_{ij} \quad (4.29)$$

for $i, j = 1, \dots, n$.

Proof. For each $i \neq j$, Theorem 4.2 in view of Table 4.1, (4.4) and (4.7) gives

$$\begin{aligned} \underline{B}_{ij} &= \min_{y \in Y} (B_y)_{ij} = \min\{m_{ij}, -m_{ij}, (2\mu_j - 1)m_{ij}, -(2\mu_j - 1)m_{ij}\} \\ &= \min\{-m_{ij}, -(2\mu_j - 1)m_{ij}\} = -m_{ij}, \end{aligned}$$

and similarly,

$$\overline{B}_{ij} = \max\{m_{ij}, (2\mu_j - 1)m_{ij}\} = m_{ij}.$$

If $i = j$, then it must be $y_i = y_j$, and hence, only the first and the last row of Table 4.1 apply, giving

$$\underline{B}_{jj} = \min_{y \in Y} (B_y)_{jj} = \min\{m_{jj}, -(2\mu_j - 1)m_{jj} + 2\mu_j\} = \min\{m_{jj}, \mu_j\} = \mu_j$$

due to (4.10) and (4.9), and similarly

$$\overline{B}_{jj} = \max\{m_{jj}, \mu_j\} = m_{jj}.$$

This proves (4.28), (4.29) and thus also (4.25), (4.26) in view of the fact that κ_j defined by (4.27) satisfies

$$-m_{jj} + \kappa_j = \mu_j$$

for each j . \square

In particular, we have the following result.

COROLLARY 4.5. *If $\rho(\Delta) < 1$, then the inverse interval matrix $[I - \Delta, I + \Delta]^{-1} = [\underline{B}, \overline{B}]$ satisfies*

$$\frac{1}{2} \leq \underline{B}_{jj} \leq 1 \leq \overline{B}_{jj}$$

for each j .

Proof. This is a consequence of Theorem 4.4 and of (4.5), (4.8). \square

Existence of explicit formulae for the inverse (4.25), (4.26) is a rare exception. The author is aware of only one another such a result, namely, if $\mathbf{A} = [\underline{A}, \overline{A}]$ satisfies $\underline{A}^{-1} \geq 0$ and $\overline{A}^{-1} \geq 0$, then

$$\mathbf{A}^{-1} = [\overline{A}^{-1}, \underline{A}^{-1}],$$

see [7]; this is a consequence of Kuttler's characterization of inverse nonnegative interval matrices in [6].

5. Attainment. According to the definition of the inverse interval matrix

$$[I - \Delta, I + \Delta]^{-1} = [\underline{B}, \overline{B}],$$

for each i, j there exists a matrix, say A^{ij} , such that $|A^{ij}| \leq \Delta$ and

$$\underline{B}_{ij} = (I - A^{ij})_{ij}^{-1}$$

(we write $(I - A^{ij})_{ij}^{-1}$ instead of $((I - A^{ij})^{-1})_{ij}$), and an analogue holds for \overline{B}_{ij} . In this section, we give an explicit expression of such an A^{ij} for each i, j .

First of all, the situation is quite evident for \overline{B} because from (4.26) we have

$$\overline{B} = M = (I - \Delta)^{-1},$$

so that all the componentwise upper bounds are attained at the inverse of $I - \Delta$. But the case of \underline{B} is more involved.

THEOREM 5.1. *For each i, j we have:*

(i) *if $i \neq j$, then*

$$\underline{B}_{ij} = (I - T_y \Delta T_y)_{ij}^{-1}$$

for each $y \in Y$ satisfying $y_i y_j = -1$,

(ii) *if $i = j$, then*

$$\underline{B}_{jj} = (I - T_y \Delta T_z)_{jj}^{-1}$$

for each $y \in Y$ satisfying $y_j = -1$ and $z = y + 2e_j$.

COMMENT 5.2. Notice that $I - T_y \Delta T_z \in [I - \Delta, I + \Delta]$ for each $y, z \in Y$.

Proof of Theorem 5.1. Let $i, j \in \{1, \dots, n\}$.

(i) For each $y \in Y$, there holds

$$(I - T_y \Delta T_y)^{-1} = (T_y (I - \Delta) T_y)^{-1} = T_y M T_y.$$

Hence, if $y_i y_j = -1$, then

$$(I - T_y \Delta T_y)_{ij}^{-1} = y_i y_j m_{ij} = -m_{ij} = \underline{B}_{ij}.$$

(ii) We have

$$\begin{aligned} I - T_y \Delta T_z &= I - T_y \Delta (T_y + 2e_j e_j^T) \\ &= (I - T_y \Delta T_y) (I - (I - T_y \Delta T_y)^{-1} 2T_y \Delta e_j e_j^T) \\ &= (I - T_y \Delta T_y) (I - 2T_y M T_y T_y \Delta e_j e_j^T) \\ &= (I - T_y \Delta T_y) (I - 2T_y M \Delta e_j e_j^T), \end{aligned}$$

so that by the Sherman-Morrison formula [12] applied to the matrix in the last parentheses,

$$\begin{aligned} (I - T_y \Delta T_z)^{-1} &= (I - 2T_y M \Delta e_j e_j^T)^{-1} (I - T_y \Delta T_y)^{-1} \\ &= \left(I + \frac{2T_y M \Delta e_j e_j^T}{1 - 2e_j^T T_y M \Delta e_j} \right) T_y M T_y \\ &= T_y M T_y + \frac{2T_y M \Delta e_j e_j^T T_y M T_y}{1 + 2(M \Delta)_{jj}} \end{aligned}$$

(because $y_j = -1$), and consequently,

$$\begin{aligned} (I - T_y \Delta T_z)_{jj}^{-1} &= m_{jj} + \frac{2(T_y M \Delta)_{jj}(T_y M T_y)_{jj}}{1 + 2(M - I)_{jj}} \\ &= m_{jj} - \frac{2(M - I)_{jj} m_{jj}}{1 + 2(M - I)_{jj}} \\ &= m_{jj} - \frac{2(m_{jj} - 1)m_{jj}}{2m_{jj} - 1} \\ &= \frac{m_{jj}}{2m_{jj} - 1} = \mu_j = \underline{B}_{jj}. \quad \square \end{aligned}$$

Hence, the \underline{B}_{ij} 's are attained at inverses of many matrices in $[I - \Delta, I + \Delta]$. But the results can be essentially simplified if we use the particular set of vectors

$$y(j) = e - 2e_j = (1, \dots, 1, -1, 1, \dots, 1)^T \quad (j = 1, \dots, n).$$

COROLLARY 5.3. *For each i, j we have:*

- (i) $\underline{B}_{ij} = (I - T_{y(j)} \Delta T_{y(j)})_{ij}^{-1}$ if $i \neq j$,
- (ii) $\underline{B}_{jj} = (I - T_{y(j)} \Delta)_{jj}^{-1}$.

Proof. The results are immediate consequences of Theorem 5.1 since $y(j)_j = -1$, $y(j)_i y(j)_j = -1$ for each $i \neq j$ and $z = y(j) + 2e_j = e$. \square

6. Properties of the matrices B_y . This section was included at a suggestion of the referee. We add here some further properties of the matrices B_y .

THEOREM 6.1. *Let $\varrho(\Delta) < 1$. Then for each $y \in Y$ the unique solution B_y of the matrix equation (4.12) satisfies:*

- (i) $B_y = I + T_y(M - I)(T_\mu + (I - T_\mu)T_y)$,
- (ii) $|B_y| = M(T_\mu + (I - T_\mu)T_y)$,
- (iii) $\| |B_y - I| - (M - I)T_\mu \| = (M - I)(I - T_\mu)$,
- (iv) $\| |B_y| - MT_\mu \| = M(I - T_\mu)$,
- (v) $T_y(B_y - I) \geq 0$,
- (vi) $|B_y| \leq M$,
- (vii) if $\Delta = pq^T$ for some $p \geq 0$ and $q \geq 0$, then

$$B_y = I + \beta T_y p q^T (T_\mu + (I - T_\mu)T_y),$$

where

$$\beta = \frac{1}{1 - q^T p}$$

and

$$\mu_j = \frac{\beta p_j q_j + 1}{2\beta p_j q_j + 1} \quad (j = 1, \dots, n).$$

Proof. (i) is a rearrangement of (4.13) based on the fact that $T_y M T_y$ can be written as $I + T_y(M - I)T_y$, and (ii) is a rearrangement of (4.24) made in order to bring the right-hand side to a form similar to that of (i).

(iii): From (i), we have

$$|B_y - I| = |T_y(M - I)(T_\mu + (I - T_\mu)T_y)| = (M - I)(T_\mu + (I - T_\mu)T_y),$$

and hence,

$$||B_y - I| - (M - I)T_\mu| = |(M - I)(I - T_\mu)T_y| = (M - I)(I - T_\mu).$$

(iv): Similarly, (ii) yields

$$||B_y| - M T_\mu| = |M(I - T_\mu)T_y| = M(I - T_\mu).$$

(v): Since B_y is a solution of the equation

$$B - T_y \Delta |B| = I, \tag{6.1}$$

we have that

$$T_y(B_y - I) = \Delta |B_y| \geq 0.$$

(vi): From (6.1) we obtain

$$|B_y| = |I + T_y \Delta |B_y|| \leq I + \Delta |B_y|,$$

and hence,

$$(I - \Delta)|B_y| \leq I.$$

Premultiplying this inequality by the nonnegative matrix $M = (I - \Delta)^{-1}$ implies

$$|B_y| \leq M.$$

(vii): Since $\varrho(\Delta) = q^T p < 1$, β is well defined and positive. The Sherman-Morrison theorem [12] then implies that

$$M = (I - p q^T)^{-1} = I + \beta p q^T$$

and

$$\mu_j = \frac{m_{jj}}{2m_{jj} - 1} = \frac{\beta p_j q_j + 1}{2\beta p_j q_j + 1} \quad (j = 1, \dots, n),$$

and the result now follows from (i). \square

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