Institute of Computer Science
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A New Characterization of the Maximum Cut in a Graph

Dedicated to the memory of the Tibetan meditation master Geshe Langri Tangpa (1054-1123), author of the "Eight verses for training the mind"

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Technical report No. V-1155

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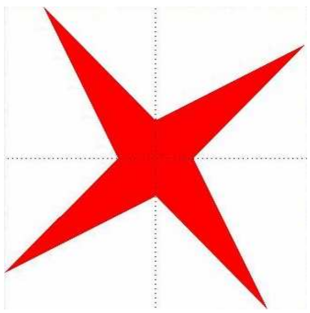
Abstract:

We prove a formula expressing the maximal cut in a graph in terms of solvability of a system of linear inequalities $-e \le Ax \le e$ (e being the vector of all ones) appended with a nonlinear constraint $\|x\|_1 \ge 1$.[2]



Keywords:
Graph, maximum cut, linear inequalities, norm.

[1]Equivalent to our "Dr".

[2]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

1 Introduction

Maximum cut in a graph is a well known NP-complete problem. In the main result of this report (Theorem 1) we prove a formula expressing the maximal cut in a graph in terms of solvability of a system of linear inequalities

$$-e \le Ax \le e$$

(e being the vector of all ones) appended with a nonlinear constraint

$$\|x\|_1 \ge 1.$$

In this way the original discrete problem is recast as a continuous weakly nonlinear problem which can be solved by nonlinear optimization techniques. A related decision problem of determining whether the maximum cut exceeds a prescribed nonnegative integer ℓ is handled in Corollary 3.

2 Maximum cut: definition

Let $G = (N, E)$ be an undirected graph with set of nodes $N = \{1, \ldots, n\}$ and set of edges E. Let m denote the cardinality of E.

Let $A_G = (a_{ij})$ be given by $a_{ij} = n$ if $i = j$, $a_{ij} = -1$ if $i \ne j$ and the nodes i, j are connected by an edge, and $a_{ij} = 0$ if $i \ne j$ and i, j are not connected. Then A_G is an MC-matrix [4].

For $S \subseteq N$, define the cut $c(S)$ as the number of edges in E whose one endpoint belongs to S and the other one to $N \setminus S$. Then the maximum cut in G is defined by

$$\mathrm{mc}(G) = \max_{S \subseteq N} c(S).$$

Computation of the maximum cut in a graph is known to be an NP-complete problem [2].

3 Maximum cut: characterization

We denote $\mathcal{N} = \{0, 1, 2, \ldots\}$ (the set of nonnegative integers), $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$, and we use the norm $\|x\|_1 = e^T|x| = \sum_{i=1}^n |x_i|$. Then we have this characterization which is the main result of this report.

Theorem 1. *For each undirected graph G there holds*

$$\mathrm{mc}(G) = \max\{\, \ell \in \mathcal{N} \mid -e \le (4\ell - 2m + n^2)A_G^{-1}x \le e,\ \|x\|_1 \ge 1 \text{ has a solution} \,\}.$$

Proof. The result follows from the relation

$$\mathrm{mc}(G) = \tfrac{1}{4}\left(\max_{z \in \{-1,1\}^n} z^T A_G z + 2m - n^2\right)$$

established in the proof of Theorem 3 in [4] and from Proposition 3 in [3]. □

It remains to be shown how a maximum cut $c(S)$ can be found.

Theorem 2. *Let x be any solution of the system*

$$-e \leq (4\mathrm{mc}(G) - 2m + n^2)A_G^{-1}x \leq e,$$

$$\|x\|_1 \geq 1.$$

Then the set

$$S = \{\, i \mid x_i \geq 0 \,\}$$

satisfies

$$c(S) = \mathrm{mc}(G).$$

Proof. This description is a consequence of construction made in the proof of Theorem 3 in [4]. □

4 Maximum cut: lower bounds

As immediate consequences of Theorems 1 and 2 we obtain these two corollaries.

Corollary 3. *Let G be an undirected graph and ℓ a nonnegative integer. Then*

$$\mathrm{mc}(G) \geq \ell \tag{4.1}$$

holds if and only if the system

$$-e \leq (4\ell - 2m + n^2)A_G^{-1}x \leq e, \tag{4.2}$$

$$\|x\|_1 \geq 1 \tag{4.3}$$

has a solution.

Corollary 4. *If the system (4.2), (4.3), where ℓ is a nonnegative integer, has a solution x, then the set*

$$S = \{\, i \mid x_i \geq 0 \,\}$$

satisfies

$$c(S) \geq \ell.$$

If (4.2), (4.3) has no solution, then

$$\mathrm{mc}(G) < \ell.$$

5 Maximum cut: algorithm

Corollary 3 shows us a way how to verify (or disprove) the inequality (4.1) via solving a system of inequalities of the type

$$-e \leq Ax \leq e, \tag{5.1}$$

$$\|x\|_1 \geq 1. \tag{5.2}$$

Such an algorithm, named **basintnpprob** [from BASic INTerval NP PROBlem], was described in [5]. As proved there, the algorithm in a finite number of steps either finds a solution to (5.1), (5.2), or states that no such solution exists.

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