Computing the Norm $\|A\|_{\infty,1}$ is NP-Hard

Dedicated to Professor Svatopluk Poljak, in memoriam

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Abstract

It is proved that computing the subordinate matrix norm $\|A\|_{\infty,1}$ is NP-hard. Even more, existence of a polynomial-time algorithm for computing this norm with relative accuracy less than $1/(4n^2)$, where n is matrix size, implies P=NP.

Key words. Norm, positive definiteness, M-matrix, NP-hardness

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1 Introduction

Given two vector norms $\|x\|_\alpha$ in R^n and $\|x\|_\beta$ in R^m, a subordinate matrix norm in $R^{m\times n}$ is defined by

$$\|A\|_{\alpha,\beta} = \max_{\|x\|_\alpha=1} \|Ax\|_\beta$$

(Golub and van Loan [4]). $\|A\|_{\alpha,\beta}$ is a matrix norm, i.e., it possesses the three usual properties: 1) $\|A\|_{\alpha,\beta} \geq 0$ and $\|A\|_{\alpha,\beta} = 0$ if and only if $A = 0$, 2) $\|A+B\|_{\alpha,\beta} \leq \|A\|_{\alpha,\beta} + \|B\|_{\alpha,\beta}$, 3) $\|\lambda A\|_{\alpha,\beta} = |\lambda|\cdot\|A\|_{\alpha,\beta}$. However, generally it does not possess the property $\|AB\|_{\alpha,\beta} \leq \|A\|_{\alpha,\beta}\|B\|_{\alpha,\beta}$ (it does e.g. if $\alpha = \beta$).

By combining the two frequently used norms

$$\|x\|_1 = \sum_i |x_i|,$$
$$\|x\|_\infty = \max_i |x_i|,$$

we get three well-known easily computable subordinate norms

$$\|A\|_{1,1} = \max_j \sum_i |a_{ij}|,$$
$$\|A\|_{\infty,\infty} = \max_i \sum_j |a_{ij}|,$$
$$\|A\|_{1,\infty} = \max_{ij} |a_{ij}|$$

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(see Golub and van Loan [4]). It turns out, however, that the fourth norm

$$\|A\|_{\infty,1} = \max_{\|x\|_\infty=1} \|Ax\|_1$$

has an exceptional behavior since it is much more difficult to compute. In this paper we prove that computing $\|A\|_{\infty,1}$ is NP-hard. For the purpose of various applications, the result is presented in several different settings (Theorems 3 to 6).

2 The norm $\|A\|_{\infty,1}$

This norm can be computed by a finite formula which, however, involves maximization over the set Z of all ± 1-vectors of length n (whose cardinality is 2^n):

Proposition 1 *For each $A \in R^{m\times n}$ we have*

$$\|A\|_{\infty,1} = \max_{z\in Z} \|Az\|_1, \tag{1}$$

where

$$Z = \{z \in R^n;\ z_j \in \{-1, 1\} \text{ for each } j\}.$$

Moreover, if A is symmetric positive semidefinite, then

$$\|A\|_{\infty,1} = \max_{z\in Z} z^T Az. \tag{2}$$

Proof. 1) If $\|x\|_\infty = 1$, then x belongs to the unit cube $\{x;\ -e \le x \le e\}$, $e = (1, \dots, 1)^T$, which is a convex polyhedron, therefore x can be expressed as a convex combination of its vertices which are exactly the points in Z:

$$x = \sum_{z\in Z} \lambda_z z, \tag{3}$$

where $\lambda_z \ge 0$ for each $z \in Z$ and $\sum_{z\in Z} \lambda_z = 1$. From (3) we have

$$\|Ax\|_1 = \|\sum_{z\in Z} \lambda_z Az\|_1 \le \max_{z\in Z} \|Az\|_1,$$

hence

$$\max_{\|x\|_\infty=1} \|Ax\|_1 \le \max_{z\in Z} \|Az\|_1 \le \max_{\|x\|_\infty=1} \|Ax\|_1$$

(since $\|z\|_\infty = 1$ for each $z \in Z$), and (1) follows.

2) Let A be symmetric positive semidefinite and let $z \in Z$. Define $y \in Z$ by $y_j = 1$ if $(Az)_j \ge 0$ and $y_j = -1$ if $(Az)_j < 0$ ($j = 1, \dots, n$), then

$$\|Az\|_1 = y^T Az.$$

Since A is symmetric positive semidefinite, we have

$$(y-z)^T A(y-z) \geq 0,$$

which implies

$$2y^T Az \leq y^T Ay + z^T Az \leq 2 \max_{z \in Z} z^T Az,$$

hence

$$\|Az\|_1 = y^T Az \leq \max_{z \in Z} z^T Az$$

and

$$\|A\|_{\infty,1} = \max_{z \in Z} \|Az\|_1 \leq \max_{z \in Z} z^T Az. \tag{4}$$

Conversely, for each $z \in Z$ we have

$$z^T Az \leq |z^T Az| \leq |z|^T |Az| = e^T |Az| = \|Az\|_1 \leq \max_{z \in Z} \|Az\|_1 = \|A\|_{\infty,1},$$

hence

$$\max_{z \in Z} z^T Az \leq \|A\|_{\infty,1},$$

which together with (4) gives (2). ■

A weaker form of (2) ($\|A\|_{\infty,1} = \max_{\|x\|_\infty = 1} x^T Ax$) was given by Tao [8]. In section 4 we shall prove that computing $\|A\|_{\infty,1}$ is NP-hard. This suggests that unless P=NP, the formulae (1), (2) cannot be essentially simplified.

3 MC-matrices

In order to prove the NP-hardness for a suitably narrow class of matrices, we introduce the following concept (first formulated in [7]):

Definition 1 *A symmetric $n \times n$ matrix $A = (a_{ij})$ is called an MC-matrix*[1] *if it is of the form*

$$a_{ij} \begin{cases} = n & \text{if } i = j, \\ \in \{0, -1\} & \text{if } i \neq j \end{cases}$$

$(i, j = 1, \ldots, n)$.

Since an MC-matrix is symmetric by definition, there are altogether $2^{n(n-1)/2}$ MC-matrices of size n. The basic properties of MC-matrices are summed up in the following proposition (where we denote, as customary, by $\|A\|_1$ the norm $\|A\|_{1,1}$ described in Section 1):

[1] from "maximum cut"; see the proof of Theorem 3 below

Proposition 2 *An MC-matrix $A \in R^{n\times n}$ is symmetric positive definite, nonnegative invertible and satisfies*

$$\|A\|_{\infty,1} = \max_{z\in Z} z^T Az, \tag{5}$$

$$n \le \|A\|_{\infty,1} \le n(2n-1) \tag{6}$$

and

$$\|A^{-1}\|_1 \le 1.$$

Proof. A is symmetric by definition; it is positive definite since for $x \ne 0$,

$$x^T Ax \ge n\|x\|_2^2 - \sum_{i\ne j} |x_i x_j| = (n+1)\|x\|_2^2 - \|x\|_1^2 \ge \|x\|_2^2 > 0$$

($\|x\|_1 \le \sqrt{n}\|x\|_2$ by the Cauchy-Schwarz inequality [4]). Hence (5) holds by Proposition 1. Since $|a_{ij}| \le 1$ for $i \ne j$, for each $z \in Z$ and each $i \in \{1, \dots, n\}$ we have

$$z_i(Az)_i = n + \sum_{j\ne i} a_{ij} z_i z_j \in [1,\, 2n-1],$$

hence

$$n \le z^T Az \le n(2n-1)$$

for each $z \in Z$, and (5) implies (6). Putting

$$A_0 = nI - A,$$

we have that $A_0 \ge 0$, $A = nI - A_0 = n(I - \frac{1}{n}A_0)$ and $\|\frac{1}{n}A_0\|_1 \le \frac{n-1}{n} < 1$, hence

$$A^{-1} = \tfrac{1}{n}\sum_0^\infty (\tfrac{1}{n}A_0)^j \ge 0$$

and

$$\|A^{-1}\|_1 \le \frac{1}{n - \|A_0\|_1} \le 1,$$

which completes the proof. ∎

Hence an MC-matrix $A \in R^{n\times n}$ satisfies

$$\|A\|_1 \cdot \|A^{-1}\|_1 < 2n,$$

i.e., it is well conditioned.

4 Computing $\|A\|_{\infty,1}$ is NP-hard

The following basic result is due to Poljak and Rohn [5] (given there in another formulation without using the concept of an MC-matrix).

Theorem 3 *The following decision problem is NP-complete:*
Instance. *An MC-matrix A and a positive integer ℓ.*
Question. *Is $z^TAz \geq \ell$ for some $z \in Z$?*

Proof. Let (N, E) be an undirected graph with $N = \{1, \dots, n\}$. Let $A = (a_{ij})$ be given by $a_{ij} = n$ if $i = j$, $a_{ij} = -1$ if $i \neq j$ and the nodes i, j are connected by an edge, and $a_{ij} = 0$ if $i \neq j$ and i, j are not connected. Then A is an MC-matrix. For $S \subseteq N$, define the cut $c(S)$ as the number of edges in E whose one endpoint belongs to S and the other one to $N - S$. We shall prove that

$$\|A\|_{\infty,1} = 4 \max_{S \subseteq N} c(S) - 2|E| + n^2 \tag{7}$$

holds, where $|E|$ denotes the cardinality of E. Given a $S \subseteq N$, define a $z \in Z$ by

$$z_i = \begin{cases} 1 & \text{if } i \in S, \\ -1 & \text{if } i \notin S. \end{cases}$$

Then we have

$$\begin{aligned} z^TAz &= \sum_{i,j} a_{ij} z_i z_j = \sum_{i \neq j} a_{ij} z_i z_j + n^2 \\ &= \sum_{i \neq j} [-\tfrac{1}{2} a_{ij} (z_i - z_j)^2 + a_{ij}] + n^2 \\ &= -\tfrac{1}{2} \sum_{z_i z_j = -1} a_{ij} (z_i - z_j)^2 + \sum_{i \neq j} a_{ij} + n^2 \\ &= -\tfrac{1}{2} \sum_{z_i z_j = -1} 4a_{ij} + \sum_{i \neq j} a_{ij} + n^2, \end{aligned}$$

hence

$$z^TAz = 4c(S) - 2|E| + n^2. \tag{8}$$

Conversely, given $z \in Z$, then for $S = \{i \in N;\ z_i = 1\}$ the same reasoning implies (8). Taking maximum on both sides of (8), we obtain (7) in view of (5).

Hence, given a positive integer L, we have that

$$c(S) \geq L \tag{9}$$

is valid for some $S \subseteq N$ if and only if

$$z^TAz \geq 4L - 2|E| + n^2$$

holds for some $z \in Z$. Since the decision problem (9) is NP-complete ("simple max-cut problem", Garey, Johnson and Stockmeyer [3]), we obtain that the decision problem

$$z^T A z \geq \ell \tag{10}$$

(ℓ positive integer) is NP-hard. It is NP-complete since for a guessed solution $z \in Z$ the validity of (10) can be checked in polynomial time. ∎

In this way, in view of (5) we have also proved the following result:

Theorem 4 *Computing $\|A\|_{\infty,1}$ is NP-hard in the class of MC-matrices.*

To facilitate formulations of some applications of these results [6], it is advantageous to remove the integer parameter ℓ from the formulation of Theorem 3. This can be done by using M-matrices instead of MC-matrices. Let us recall that $A = (a_{ij})$ is called an M-matrix if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$ (a number of equivalent formulations can be found in Berman and Plemmons [1]); hence each MC-matrix is an M-matrix (Proposition 2). Since a symmetric M-matrix is positive definite [1], this property is not explicitly mentioned in the following theorem:

Theorem 5 *The following decision problem is NP-hard:*
Instance. *A symmetric rational M-matrix A.*
Question. *Is $\|A\|_{\infty,1} \geq 1$?*

Proof. Given an MC-matrix A and a positive integer ℓ, the assertion

$$z^T A z \geq \ell \text{ for some } z \in Z$$

is equivalent to $\|A\|_{\infty,1} \geq \ell$ and thereby also to

$$\left\| \tfrac{1}{\ell} A \right\|_{\infty,1} \geq 1,$$

where $\frac{1}{\ell} A$ is a symmetric rational M-matrix. Hence the decision problem of Theorem 3 can be reduced in polynomial time to the current one, which is then NP-hard. ∎

Finally we shall show that even computing a sufficiently close approximation of $\|A\|_{\infty,1}$ is NP-hard:

Theorem 6 *Suppose there exists a polynomial-time algorithm which for each MC-matrix A computes a rational number $\nu(A)$ satisfying*

$$\left| \frac{\nu(A) - \|A\|_{\infty,1}}{\|A\|_{\infty,1}} \right| \leq \frac{1}{4n^2},$$

where n is the size of A. Then P=NP.

Proof. If such an algorithm exists, then

$$|\nu(A) - \|A\|_{\infty,1}| \leq \frac{\|A\|_{\infty,1}}{4n^2} \leq \frac{n(2n-1)}{4n^2} < \frac{1}{2}$$

due to (6), which implies

$$\|A\|_{\infty,1} < \nu(A) + \tfrac{1}{2} < \|A\|_{\infty,1} + 1,$$

and thereby also

$$\|A\|_{\infty,1} = \left\lfloor \nu(A) + \tfrac{1}{2} \right\rfloor$$

(since $\|A\|_{\infty,1}$ is integer for an MC-matrix A by (5)). Hence the NP-hard problem of Theorem 4 can be solved in polynomial time, implying P=NP. ∎

Various applications of Theorem 5 for problems with inexact data (regularity, positive definiteness, stability, solvability of linear equations and inequalities, linear and quadratic programming) are given in [6].

5 Concluding remarks

We have proved that existence of a polynomial-time algorithm for computing $\|A\|_{\infty,1}$ with relative accuracy less than $\frac{1}{4n^2}$ implies that the complexity classes P and NP are equal. This runs against the famous unproved *conjecture* that P≠NP holds, which is widely believed to be true (see Garey and Johnson [2] for details). Hence, the existence of such a polynomial-time algorithm seems highly unlikely, although it cannot be ruled out by current results in complexity theory.

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