Perron Vectors of an Irreducible Nonnegative Interval Matrix

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Abstract

As is well known, an irreducible nonnegative matrix possesses a uniquely determined Perron vector. As the main result of this paper we give a description of the set of Perron vectors of all the matrices contained in an irreducible nonnegative interval matrix $\mathbf{A}$. This result is then applied to show that there exists a subset $\mathbf{A}_*$ of $\mathbf{A}$ parameterized by n parameters (instead of n^2 ones in the description of $\mathbf{A}$) such that for each $A \in \mathbf{A}$ there exists a matrix $A' \in \mathbf{A}_*$ having the same spectral radius and the same Perron vector as A.

Key words. Nonnegative matrix, irreducible matrix, interval matrix, Perron vector.

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1 Irreducible interval matrices

In this paper we consider only square $n \times n$ matrices. Such a matrix A is called nonnegative if all its entries are nonnegative. A nonnegative matrix $A \in \mathbb{R}^{n\times n}$ is said to be reducible if there exists a permutation matrix P such that

$$P^T A P = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where B and D are square matrices (i.e., at least of size 1×1), and it is called irreducible if it is not reducible. The basic eigenvalue properties of irreducible nonnegative matrices are summed up in the Perron-Frobenius theorem (see Horn and Johnson [3], p. 508). We formulate here only a portion of it relevant to the scope of this paper; $\varrho(A)$ denotes the spectral radius of A, $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$, and $x > 0$ means that all entries of x are positive.

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Theorem 1. *For each irreducible nonnegative matrix A there exists a unique vector x satisfying*

$$\begin{aligned} Ax &= \varrho(A)x, && (1)\\ e^T x &= 1, && (2)\\ x &> 0, && (3) \end{aligned}$$

and no eigenvalue $\lambda \neq \varrho(A)$ has a positive eigenvector.

The positive eigenvector determined uniquely by (1)–(3) is called the Perron vector of A; we shall denote it by $x(A)$.

Given $\underline{A}$, $\overline{A} \in \mathbb{R}^{n\times n}$ with $\underline{A} \le \overline{A}$, the set

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{ A \mid \underline{A} \le A \le \overline{A} \}$$

is called an interval matrix with the bounds $\underline{A}$ and $\overline{A}$ (see e.g. Neumaier [4] for basic facts concerning interval matrices). $\mathbf{A}$ is said to be nonnegative if $\underline{A} \ge 0$, which is the same as to say that all matrices in $\mathbf{A}$ are nonnegative. A nonnegative interval matrix $\mathbf{A}$ is called irreducible if each $A \in \mathbf{A}$ is irreducible. It turns out that checking irreducibility of $\mathbf{A} = [\underline{A}, \overline{A}]$ reduces to checking this property for $\underline{A}$ only. The following proposition is a consequence of a more general result (Berman and Plemmons [1], Corollary 1.3.21), but we include an elementary proof of it for the sake of completeness.

Proposition 2. *A nonnegative interval matrix $[\underline{A}, \overline{A}]$ is irreducible if and only if $\underline{A}$ is irreducible.*

Proof. If each $A \in [\underline{A}, \overline{A}]$ is irreducible, then so is $\underline{A}$. Conversely, assume that $\underline{A}$ is irreducible and that some $A \in [\underline{A}, \overline{A}]$ is reducible, so that there exists a permutation matrix P such that

$$P^T A P = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where 0 is of size at least 1×1. Then from $0 \le \underline{A} \le A$ it follows

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \le P^T \underline{A} P = \begin{pmatrix} B_1 & C_1 \\ E_1 & D_1 \end{pmatrix} \le P^T A P = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

which implies that $E_1 = 0$, hence $\underline{A}$ is reducible. This contradiction shows that each $A \in [\underline{A}, \overline{A}]$ is irreducible, and the proof is complete. □

In this paper we are interested in description of the set of Perron vectors of all the matrices contained in a given irreducible nonnegative interval matrix $\mathbf{A}$. As far as we know, this topic has not been studied yet.

2 Perron vectors of an interval matrix

The set of spectral radii of all the matrices contained in an irreducible nonnegative interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ is easy to describe:

$$\{ \varrho(A) \mid A \in \mathbf{A} \} = [\varrho(\underline{A}),\ \varrho(\overline{A})],$$

because the spectral radius is a continuous function of A (Horn and Johnson [3], p. 313), hence the real function $\psi(t) = \varrho(\underline{A} + t(\overline{A} - \underline{A}))$ is continuous in $[0, 1]$, so that it attains all the intermediate values between the endpoint values $\varrho(\underline{A})$ and $\varrho(\overline{A})$, and no spectral radius can exceed this interval because $0 \le \underline{A} \le A \le \overline{A}$ implies that $\varrho(\underline{A}) \le \varrho(A) \le \varrho(\overline{A})$ (Horn and Johnson [3], p. 491).

The following main result of this paper presents a description of the set $\{ x(A) \mid A \in \mathbf{A} \}$ of the Perron vectors of all matrices contained in a given irreducible nonnegative interval matrix $\mathbf{A}$.

Theorem 3. *Let $\mathbf{A} = [\underline{A}, \overline{A}]$ be an irreducible nonnegative interval matrix. Then a vector $x \in \mathbb{R}^n$ is the Perron vector of some matrix $A \in \mathbf{A}$ if and only if it satisfies*

$$\underline{A}xx^T \le xx^T\overline{A}^T, \quad (4)$$
$$e^T x = 1, \quad (5)$$
$$x > 0. \quad (6)$$

Proof. Let x be the Perron vector of some matrix $A \in [\underline{A}, \overline{A}]$, so that (1)–(3) hold. Then from $\underline{A} \le A \le \overline{A}$ in view of positivity of x we obtain

$$\underline{A}x \le Ax = \varrho(A)x \le \overline{A}x,$$

hence for each $i, j = 1, \ldots, n$ we have

$$\frac{(\underline{A}x)_i}{x_i} \le \varrho(A) \le \frac{(\overline{A}x)_j}{x_j}$$

and thus also

$$(\underline{A}xx^T)_{ij} = (\underline{A}x)_i x_j \le x_i(\overline{A}x)_j = (xx^T\overline{A}^T)_{ij},$$

which proves (4); (5) and (6) are given by (2), (3).

Conversely, let x satisfy (4)–(6). Then for each i, j we have

$$(\underline{A}x)_i x_j = (\underline{A}xx^T)_{ij} \le (xx^T\overline{A}^T)_{ij} = x_i(\overline{A}x)_j,$$

hence

$$\frac{(\underline{A}x)_i}{x_i} \le \frac{(\overline{A}x)_j}{x_j},$$

which implies that

$$\max_i \frac{(\underline{A}x)_i}{x_i} \le \min_j \frac{(\overline{A}x)_j}{x_j}.$$

Let us choose any λ satisfying

$$\max_i \frac{(\underline{A}x)_i}{x_i} \le \lambda \le \min_j \frac{(\overline{A}x)_j}{x_j}.$$

Then from the first inequality it follows that $\underline{A}x \le \lambda x$, whereas the second one gives $\lambda x \le \overline{A}x$, together

$$\underline{A}x \le \lambda x \le \overline{A}x. \tag{7}$$

For each $i = 1, \ldots, n$ define a real function of one real variable t by

$$\varphi_i(t) = ((\underline{A} + t(\overline{A} - \underline{A}))x - \lambda x)_i.$$

Then $\varphi_i(0) = (\underline{A}x - \lambda x)_i \le 0$ and $\varphi_i(1) = (\overline{A}x - \lambda x)_i \ge 0$ by (7), hence by continuity of φ_i there exists a $t_i \in [0, 1]$ such that $\varphi_i(t_i) = 0$. Now put

$$A = \underline{A} + \operatorname{diag}(t_1, \ldots, t_n)(\overline{A} - \underline{A})$$

(where $\operatorname{diag}(t_1, \ldots, t_n)$ denotes the diagonal matrix with diagonal entries $t_1, \ldots, t_n$), then $A \in [\underline{A}, \overline{A}]$ because $t_i \in [0, 1]$ for each i, and we have $(Ax - \lambda x)_i = \varphi_i(t_i) = 0$ for each i, hence

$$Ax = \lambda x.$$

Since $e^T x = 1$ and $x > 0$ by (5), (6), Theorem 1 gives that $\lambda = \varrho(A)$ and $x = x(A)$, hence x is the Perron vector of A, which proves the second implication. □

The inequality (4) could also be written in a more "symmetric" form

$$\underline{A}xx^T \le (\overline{A}xx^T)^T,$$

but we prefer the form (4) which, as we have seen, arises naturally in the proof.

The construction given in the second part of the proof is worth summarizing as a separate assertion.

Theorem 4. *Let x satisfy (4)–(6). Then*

$$\max_i \frac{(\underline{A}x)_i}{x_i} \le \min_j \frac{(\overline{A}x)_j}{x_j} \tag{8}$$

and for each λ with

$$\max_i \frac{(\underline{A}x)_i}{x_i} \le \lambda \le \min_j \frac{(\overline{A}x)_j}{x_j} \tag{9}$$

there holds $\lambda = \varrho(A)$ and $x = x(A)$, where the matrix $A \in [\underline{A}, \overline{A}]$ is given by

$$A = \underline{A} + \operatorname{diag}(t_1, \ldots, t_n)(\overline{A} - \underline{A}), \tag{10}$$

with

$$t_i = \begin{cases} (\lambda x - \underline{A}x)_i/((\overline{A} - \underline{A})x)_i & \text{if } ((\overline{A} - \underline{A})x)_i > 0, \\ 1 & \text{if } ((\overline{A} - \underline{A})x)_i = 0 \end{cases} \qquad (i = 1, \ldots, n). \tag{11}$$

Proof. As everything else has been stated in the proof of Theorem 3, it remains to explain the formula (11) for t_i only. This value is a solution of the equation $\varphi_i(t_i) = 0$, i.e., it satisfies

$$t_i((\overline{A} - \underline{A})x)_i = (\lambda x - \underline{A}x)_i. \tag{12}$$

If $((\overline{A} - \underline{A})x)_i > 0$, then this equation has the unique solution

$$t_i = \frac{(\lambda x - \underline{A}x)_i}{((\overline{A} - \underline{A})x)_i}.$$

If $((\overline{A} - \underline{A})x)_i = 0$, then, since we know from the proof of Theorem 3 that equation (12) has a solution, it must be $(\lambda x - \underline{A}x)_i = 0$, hence the equation is satisfied for any $t_i \in \mathbb{R}$, thus also for our choice $t_i = 1$. □

3 The subset $\mathbf{A}_*$

In accordance with the construction made in (10), denote

$$\mathbf{A}_* = \{\, \underline{A} + T(\overline{A} - \underline{A}) \mid 0 \le T \le I \,\},$$

so that $\mathbf{A}_*$ is a subset of $\mathbf{A}$. Let us compare it with the description of $\mathbf{A}$ which can also be written as

$$\mathbf{A} = \{\, \underline{A} + T(\overline{A} - \underline{A}) \mid 0 \le T \le ee^T \}.$$

We can see that the description of $\mathbf{A}_*$ involves n "parameters" $t_{ii} \in [0,1]$ $(i = 1, \ldots, n)$, whereas that of $\mathbf{A}$ contains n^2 "parameters" $t_{ij} \in [0,1]$ $(i, j = 1, \ldots, n)$. Nevertheless, the following consequence of Theorem 4 shows that all the spectral radii and Perron vectors of $\mathbf{A}$ are attained over its subset $\mathbf{A}_*$.

Theorem 5. *Let $\mathbf{A}$ be an irreducible nonnegative interval matrix. Then for each $A \in \mathbf{A}$ there exists an $A' \in \mathbf{A}_*$ such that $\varrho(A) = \varrho(A')$ and $x(A) = x(A')$.*

Proof. Let $A \in \mathbf{A}$. Then $x = x(A)$ satisfies (4)–(6) by Theorem 3 and there holds

$$\varrho(A) = \frac{(Ax)_k}{x_k}$$

for each k, so that from $\underline{A} \le A \le \overline{A}$ it follows

$$\frac{(\underline{A}x)_k}{x_k} \le \varrho(A) \le \frac{(\overline{A}x)_k}{x_k}$$

for each k, hence $\lambda = \varrho(A)$ satisfies (9) and a direct application of Theorem 4 gives that $\varrho(A) = \varrho(A')$ and $x(A) = x(A')$, where A' is given by (10), (11) and thus belongs to $\mathbf{A}_*$. □

Finally we note that eigenvectors of interval matrices were examined from another point of view by Hartfiel [2].

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References

[1] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia, 1994.

[2] D. J. Hartfiel, *Eigenvector sets for intervals of matrices*, Linear Algebra and Its Applications, 262 (1997), pp. 1–10.

[3] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.

[4] A. Neumaier, *Interval Methods for Systems of Equations*, Cambridge University Press, Cambridge, 1990.