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A Perturbation Theorem for Linear Equations

Jiří Rohn

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Pod Vodárenskou věží 2, 182 07 Prague 8, phone: +420 266 051 111, fax: +420 286 585 789,
e-mail:rohn@cs.cas.cz

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Abstract:

This is an unpublished two-page manuscript from 2000. We describe explicit formulae for componentwise bounds on solution of a system of linear equations $A_c x = b_c$ (A_c square) under perturbation of all data. To make the result numerically tractable, we avoid use of exact inverses, using instead some matrices R and M required only to satisfy certain inequalities. Hansen's optimality result is a special case of our theorem.

Keywords:
Linear equations, perturbation, bounds.

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Rohn, J.

A perturbation theorem for linear equations

We describe explicit formulae for componentwise bounds on solution of a system of linear equations $A_c x = b_c$ (A_c square) under perturbation of all data. To make the result numerically tractable, we avoid use of exact inverses, using instead some matrices R and M required only to satisfy certain inequalities. Hansen's optimality result is a special case of our theorem.

Notations used: I is the unit matrix, ϱ denotes the spectral radius, for $A = (a_{ij})$ we denote $|A| = (|a_{ij}|)$ and inequalities are understood componentwise. To save space, we write a/b instead of $\frac{a}{b}$.

Theorem. *Let $M \geq 0$ and R be arbitrary matrices satisfying*

$$MG + I \leq M, \tag{1}$$

where $G = |I - RA_c| + |R|\Delta$. Then for each A and b such that $|A - A_c| \leq \Delta$ and $|b - b_c| \leq \delta$, A is nonsingular and the solution of the system $Ax = b$ satisfies for each $i \in \{1, \ldots, n\}$

$$\min\{\underset{\sim}{x}_i/\alpha_i,\ \underset{\sim}{x}_i/\beta_i\} \leq x_i \leq \max\{\tilde{x}_i/\alpha_i,\ \tilde{x}_i/\beta_i\}, \tag{2}$$

where

$$\begin{aligned}
\underset{\sim}{x}_i &= -(M(|Rb_c| + |R|\delta))_i + m_i(Rb_c + |Rb_c|)_i \\
\tilde{x}_i &= (M(|Rb_c| + |R|\delta))_i + m_i(Rb_c - |Rb_c|)_i \\
\alpha_i &= 1 + (|r_i| - r_i)m_i + h_i \\
\beta_i &= 2m_i - 1 - (|r_i| + r_i)m_i - h_i \\
m_i &= M_{ii} \\
r_i &= (I - RA_c)_{ii} \\
h_i &= (M - MG - I)_{ii}
\end{aligned}$$

and $\beta_i \geq \alpha_i \geq 1$. Moreover, if $A_c = I$ and $\varrho(\Delta) < 1$ and if we take $R = I$ and $M = (I - \Delta)^{-1}$, then the bounds (2) are exact (i.e., attained).

Proof. 1) First we prove that each matrix A with $|A - A_c| \leq \Delta$ is nonsingular. Premultiplying the inequality (1) by the nonnegative matrix G yields $MG^2 + G + I \leq MG + I \leq M$ and by induction $\sum_{j=0}^{k} G^j \leq MG^{k+1} + \sum_{j=0}^{k} G^j \leq M$ for each $k \geq 0$, hence $\sum_0^\infty G^j$ is convergent which, as well known, implies that $\varrho(G) < 1$. Since $|I - RA| = |I - RA_c + R(A_c - A)| \leq |I - RA_c| + |R|\Delta = G$, we have $\varrho(I - RA) \leq \varrho(G) < 1$ which means that the matrix $RA = I - (I - RA)$ is nonsingular, hence A is nonsingular.

2) Next we prove that $\beta_i \geq \alpha_i \geq 1$ for each i. From the definition of h_i we have $m_i = (MG)_{ii} + 1 + h_i \geq m_i|r_i| + 1 + h_i$ which can be easily rearranged to $2m_i - 1 - (|r_i| + r_i)m_i - h_i \geq 1 + (|r_i| - r_i)m_i + h_i$, giving $\beta_i \geq \alpha_i$; the inequality $\alpha_i \geq 1$ follows from the nonnegativity of m_i and h_i.

3) Let x solve $Ax = b$ for some A, b with $|A - A_c| \leq \Delta$ and $|b - b_c| \leq \delta$. Then we have

$$x = (I - RA)x + Rb = (I - RA_c)x + R(A_c - A)x + Rb_c + R(b - b_c) \tag{3}$$

and taking absolute values gives

$$|x| \leq G|x| + |Rb_c| + |R|\delta. \tag{4}$$

Let $i \in \{1, \ldots, n\}$. Then from the ith equation in (3) we have

$$\begin{aligned}
x_i &\leq ((I - RA_c)x)_i + (|R|\Delta|x|)_i + (Rb_c)_i + (|R|\delta)_i \\
&= (G|x| + |Rb_c| + |R|\delta)_i + ((I - RA_c)x - |I - RA_c| \cdot |x| + Rb_c - |Rb_c|)_i.
\end{aligned} \tag{5}$$

Put $x' = (|x_1|, \dots, |x_{i-1}|, x_i, |x_{i+1}|, \dots, |x_n|)^T$. Then from (4) and (5) we have $x' \le G|x| + |Rb_c| + |R|\delta + ((I - RA_c)x - |I - RA_c| \cdot |x| + Rb_c - |Rb_c|)_i e_i$, where e_i is the ith column of I. Premultiplying this inequality by the nonnegative vector $e_i^T M$ and using the matrix $H := M - MG - I$, we obtain $(Mx')_i \le ((M - H - I)|x|)_i + ((I - RA_c)x - |I - RA_c| \cdot |x|)_i m_i + \tilde{x}_i$ and consequently

$$(M(x' - |x|))_i + (H|x|)_i + |x_i| + (|I - RA_c| \cdot |x| - (I - RA_c)x)_i m_i \le \tilde{x}_i. \tag{6}$$

Since $(M(x' - |x|))_i = m_i(x_i - |x_i|)$, $(H|x|)_i \ge h_i|x_i|$ and $(|I - RA_c| \cdot |x| - (I - RA_c)x)_i \ge |r_i| \cdot |x_i| - r_i x_i$, from (6) we finally obtain an inequality containing variable x_i only:

$$m_i(x_i - |x_i|) + h_i|x_i| + |x_i| + (|r_i| \cdot |x_i| - r_i x_i)m_i \le \tilde{x}_i. \tag{7}$$

If $x_i \ge 0$, then this inequality becomes $\alpha_i x_i \le \tilde{x}_i$, implying $x_i \le \tilde{x}_i/\alpha_i$, and if $x_i < 0$, then (7) turns into $\beta_i x_i \le \tilde{x}_i$, giving $x_i \le \tilde{x}_i/\beta_i$, which together yields

$$x_i \le \max\{\tilde{x}_i/\alpha_i,\, \tilde{x}_i/\beta_i\}. \tag{8}$$

In this way we have obtained the upper bound in (2). To prove the lower one, notice that $-x$ satisfies $A(-x) = -b$, where $|A - A_c| \le \Delta$ and $|(-b) - (-b_c)| \le \delta$. Hence we can use the previously obtained result if we set $b_c := -b_c$, which affects $\tilde{x}_i$ only. Then from (8) we get $-x_i \le \max\{-\underset{\sim}{x}_i/\alpha_i,\, -\underset{\sim}{x}_i/\beta_i\}$ which, after premultiplying by -1, gives the lower bound in (2).

4) Finally, to prove the optimality result for the case $A_c = I$ and $\varrho(\Delta) < 1$, take $R = I$ and $M = (I - \Delta)^{-1}$, then $M \ge 0$, $G = \Delta$ and (1) is satisfied as an equation; moreover, for each i we have $r_i = h_i = 0$, $\alpha_i = 1$, $\beta_i = 2m_i - 1$, hence (2) has the form

$$\min\{\underset{\sim}{x}_i,\, \underset{\sim}{x}_i/\beta_i\} \le x_i \le \max\{\tilde{x}_i,\, \tilde{x}_i/\beta_i\}. \tag{9}$$

To prove that the upper bound is really attained, let us fix an $i \in \{1, \dots, n\}$ and define a diagonal matrix D by $D_{jj} = 1$ if $j \ne i$ and $(b_c)_j \ge 0$, $D_{jj} = -1$ if $j \ne i$ and $(b_c)_j < 0$, and $D_{jj} = 1$ if $j = i$, and let $\tilde{b} = Db_c + \delta$. Then it can be easily verified that $\tilde{x}_i = (M\tilde{b})_i$ holds. First, define $x' = DM\tilde{b}$. Since $M = (I - \Delta)^{-1}$ implies $\Delta M = M\Delta = M - I$, we have $(I - D\Delta D)x' = DM\tilde{b} - D(M - I)\tilde{b} = D\tilde{b} = b_c + D\delta$, which means that x' solves the system $(I - D\Delta D)x' = b_c + D\delta$ where $|(I - D\Delta D) - I| = \Delta$, $|(b_c + D\delta) - b_c| = \delta$ and $x'_i = e_i^T DM\tilde{b} = e_i^T M\tilde{b} = (M\tilde{b})_i = \tilde{x}_i$, which shows that $\tilde{x}_i$ is attained. Second, let $x'' = DM(\tilde{b} - 2(\tilde{x}_i/\beta_i)\Delta e_i)$ and define a diagonal matrix D' by $D'_{ii} = -1$ and $D'_{jj} = D_{jj}$ otherwise. Then $(I - D\Delta D')DM = DM - D\Delta(I - 2e_ie_i^T)M = DM - D(M - I) + 2D\Delta e_ie_i^T M = D + 2D\Delta e_ie_i^T M$, hence $(I - D\Delta D')x'' = (D + 2D\Delta e_ie_i^T M)(\tilde{b} - 2(\tilde{x}_i/\beta_i)\Delta e_i) = D\tilde{b} + 2\tilde{x}_i D\Delta e_i(-(1/\beta_i) + 1 - (2/\beta_i)(m_i - 1)) = D\tilde{b} = b_c + D\delta$, which shows that x'' is a solution to the system $(I - D\Delta D')x'' = b_c + D\delta$ where $|(I - D\Delta D') - I| = \Delta$, $|(b_c + D\delta) - b_c| = \delta$ and $x''_i = e_i^T DM(\tilde{b} - 2(\tilde{x}_i/\beta_i)\Delta e_i) = \tilde{x}_i - 2(\tilde{x}_i/\beta_i)(m_i - 1) = \tilde{x}_i/\beta_i$. This proves that $\tilde{x}_i/\beta_i$ is attained, hence also the upper bound $\max\{\tilde{x}_i,\, \tilde{x}_i/\beta_i\}$ in (9) is attained. The proof for the lower bound follows from the result just obtained by applying it to the case $b_c := -b_c$ as in the part 3). □

The quantities r_i and h_i correct the influence of the approximate inverses R and M; they vanish if $R = A_c^{-1}$ and $M = (I - G)^{-1} \ge 0$ are used. The last statement of the theorem is Hansen's optimality result [1] as reformulated in [2]. Matrices R and $M \ge 0$ satisfying (1) exist if and only if $\varrho(|A_c^{-1}|\Delta) < 1$ holds (Theorem 1 in [3]).

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Addresses: Jiří Rohn, Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 118 00 Prague, and Institute of Computer Science, Academy of Sciences, Pod vodárenskou věží 2, 182 07 Prague, Czech Republic.