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Radii of solvability and unsolvability of linear systems

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ABSTRACT

We consider a problem of determining the component-wise distance (called the radius) of a linear system of equations or inequalities to a system that is either solvable or unsolvable. We propose explicit characterization of these radii and show relations between them. Then the radii are classified in the polynomial vs. NP-hard manner. We also present a generalization to an arbitrary linear system consisting from both equations and inequalities with both free and nonnegative variables. Eventually, we extend the concept of the component-wise distance to a non-uniform one.

1. Introduction

We start with two motivation problems:

- *Total least squares.* An overdetermined system $Ax = b$ is typically unsolvable, and the total least square solution is a solution of $(A + A')x = b + b'$, where $(A' \mid b')$ is

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minimized in some matrix norm. Usually, the Frobenius norm is utilized, however, other norms can be employed as well [1,2]. For instance, [3] used the Chebyshev norm.

- *Radius of nonsingularity.* Given $A \in \mathbb{R}^{n\times n}$, we ask what is the distance to the nearest singular matrix. Herein, usually the Chebyshev norm is considered [4–6].

The common denominator of these two problems is that, given a linear algebraic problem, we want to find the minimal perturbation of data (in some matrix norm) such that the problem satisfies some property. In this paper, we focus on a very basic property of solvability and unsolvability of linear systems of equations and inequalities subject to perturbations with minimal Chebyshev norm.

Notation. The Chebyshev (maximum) matrix norm of A is $\|A\|_{1,\infty} := \max_{i,j} |a_{i,j}|$. Further, E and e stand for the matrix and the vector of ones, respectively, and e_k for the kth canonical unit vector. For a matrix A, we use A_{i*} and A_{*j} to denote its ith row and jth column, respectively, and $r^+ := \max\{0, r\}$ denotes the positive part of a real r. We say that a system of equations or inequalities is *solvable* if it has a solution, and it is *feasible* if it has a non-negative solution.

Problem formulation. In this paper, we will deal with the following radii of (un)solvability; in particular, we discuss their explicit characterization, computational complexity and other properties.

Definition 1. For a system $Ax = b$ we introduce the radii of solvability as follows

$$r^u_= := \inf\{\|(A' \mid b')\|_{1,\infty};\ (A + A')x = b + b' \text{ is solvable}\},$$
$$r^s_= := \inf\{\|(A' \mid b')\|_{1,\infty};\ (A + A')x = b + b' \text{ is unsolvable}\},$$
$$r^i_= := \inf\{\|(A' \mid b')\|_{1,\infty};\ (A + A')x = b + b' \text{ is feasible}\},$$
$$r^f_= := \inf\{\|(A' \mid b')\|_{1,\infty};\ (A + A')x = b + b' \text{ is infeasible}\}.$$

For a system $Ax \le b$ we introduce the radii of solvability as follows

$$r^u_\le := \inf\{\|(A' \mid b')\|_{1,\infty};\ (A + A')x \le b + b' \text{ is solvable}\},$$
$$r^s_\le := \inf\{\|(A' \mid b')\|_{1,\infty};\ (A + A')x \le b + b' \text{ is unsolvable}\},$$
$$r^i_\le := \inf\{\|(A' \mid b')\|_{1,\infty};\ (A + A')x \le b + b' \text{ is feasible}\},$$
$$r^f_\le := \inf\{\|(A' \mid b')\|_{1,\infty};\ (A + A')x \le b + b' \text{ is infeasible}\}.$$

Preliminaries. Our approach is mostly based on interval computation. By an interval matrix we mean a family of matrices

$$\boldsymbol{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n};\ \underline{A} \le A \le \overline{A}\},$$

where $\underline{A} \leq \overline{A}$ are given and inequalities between matrices are understood entrywise. The midpoint and radius matrices of $\boldsymbol{A}$ are defined as

$$A^c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A^\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

Interval vectors are defined and denoted accordingly. An interval linear system of equations, denoted briefly as $\boldsymbol{A}x = \boldsymbol{b}$, is a family of linear systems

$$Ax = b, \quad A \in \boldsymbol{A},\ b \in \boldsymbol{b}.$$

A solution of this interval system is a solution of any linear system belonging to this family. Correspondingly, we define an interval system of inequalities and their solutions.

Solutions of interval equations are characterized by the Oettli–Prager theorem [7] and solutions of interval inequalities by the Gerlach theorem [8]. For corollaries and generalization see [9,10].

Theorem 1 *(Oettli–Prager, 1964). A vector $x \in \mathbb{R}^n$ is a solution of $\boldsymbol{A}x = \boldsymbol{b}$ if and only if*

$$|A^c x - b^c| \leq A^\Delta |x| + b^\Delta. \tag{1}$$

Theorem 2 *(Gerlach, 1981). A vector $x \in \mathbb{R}^n$ is a solution of $\boldsymbol{A}x \leq \boldsymbol{b}$ if and only if*

$$A^c x \leq A^\Delta |x| + \overline{b}. \tag{2}$$

We will also utilize Farkas lemma in several forms; see [9]. In fact, Lemma 1 is not a Farkas-type statement since it considers only linear equations, but thematically it belongs to these results.

Lemma 1. *A system $Ax = b$ is unsolvable if and only if the system*

$$A^T y = 0,\ b^T y = -1$$

is solvable.

Lemma 2. *A system $Ax = b$ is infeasible if and only if the system*

$$A^T y \geq 0,\ b^T y \leq -1$$

is solvable.

Lemma 3. *A system $Ax \leq b$ is unsolvable if and only if the system*

$$A^T y = 0,\ b^T y = -1$$

is feasible.

Lemma 4. *A system $Ax \le b$ is infeasible if and only if the system*

$$A^T y \ge 0,\ \ b^T y \le -1$$

is feasible.

2. Results

2.1. Characterization

Theorem 3. *We have*

$$\begin{aligned} r^u_= &= \min_x \frac{\|Ax-b\|_\infty}{\|x\|_1+1},\\ r^s_= &= \min_{y\neq 0} \frac{\|(y^TA, y^Tb+1)\|_\infty}{\|y\|_1},\\ r^i_= &= \min_{x\ge 0} \frac{\|Ax-b\|_\infty}{\|x\|_1+1},\\ r^f_= &= \min_{y\neq 0} \frac{\|(-y^TA, y^Tb+1)^+\|_\infty}{\|y\|_1}. \end{aligned}$$

Proof. In view of Theorem 1, the value of $r^u_=$ can be expressed as

$$r^u_= = \inf\{\delta \ge 0;\ |Ax-b| \le \delta E|x| + \delta e \text{ is solvable}\},$$

from which

$$r^u_= = \min_x \max_i \frac{|Ax-b|_i}{e^T|x|+1} = \min_x \frac{\|Ax-b\|_\infty}{\|x\|_1+1}.$$

In view of Theorem 1 and Lemma 1, the value of $r^s_=$ can be expressed as

$$r^s_= = \inf\{\delta \ge 0;\ |A^Ty| \le \delta E|y|,\ \ |b^Ty+1| \le \delta e^T|y| \text{ is solvable}\},$$

from which

$$r^s_= = \min_{y\neq 0} \frac{\max\{\|A^Ty\|_\infty, |b^Ty+1|\}}{e^T|y|} = \min_{y\neq 0} \frac{\|(y^TA, y^Tb+1)\|_\infty}{\|y\|_1}.$$

In view of Theorem 1, the value of $r^i_=$ can be expressed as

$$r^i_= = \inf\{\delta \ge 0;\ |Ax-b| \le \delta E|x| + \delta e,\ \text{is feasible}\}, \tag{3}$$

from which

$$r^i_= = \min_{x\geq 0} \max_i \frac{|Ax-b|_i}{e^T|x|+1} = \min_{x\geq 0} \frac{\|Ax-b\|_\infty}{\|x\|_1+1}.$$

In view of Theorem 2 and Lemma 2, the value of $r^f_=$ can be expressed as

$$r^f_= = \inf\{\delta\geq 0;\ -A^Ty \leq \delta E|y|,\ b^Ty \leq \delta e^T|y| - 1 \text{ is solvable}\},$$

from which

$$\begin{aligned} r^f_= &= \min_{y\neq 0} \frac{\max\{0, \max_i(-A^Ty)_i, b^Ty+1\}}{e^T|y|} \\ &= \min_{y\neq 0} \frac{\|(-y^TA, y^Tb+1)^+\|_\infty}{\|y\|_1}. \quad \square \end{aligned}$$

Notice that the expression for $r^u_=$ already appeared in [3].

Theorem 4. *We have*

$$\begin{aligned} r^u_\leq &= \min_x \frac{\|(Ax-b)^+\|_\infty}{\|x\|_1+1}, \\ r^s_\leq &= \min_{y\geq 0,\ y\neq 0} \frac{\|(y^TA, y^Tb+1)\|_\infty}{\|y\|_1}, \\ r^i_\leq &= \min_{x\geq 0} \frac{\|(Ax-b)^+\|_\infty}{\|x\|_1+1}, \\ r^f_\leq &= \min_{y\geq 0,\ y\neq 0} \frac{\|(-y^TA, y^Tb+1)^+\|_\infty}{\|y\|_1}. \end{aligned}$$

Proof. In view of Theorem 2, the value of $r^u_\leq$ can be expressed as

$$r^u_\leq = \inf\{\delta\geq 0;\ Ax-b\leq \delta E|x| + \delta e \text{ is solvable}\},$$

from which

$$r^u_\leq = \min_x \frac{\|(Ax-b)^+\|_\infty}{\|x\|_1+1}.$$

In view of Theorem 1 and Lemma 3, the value of $r^s_\leq$ can be expressed as

$$r^s_\leq = \inf\{\delta\geq 0;\ |A^Ty|\leq \delta E|y|,\ |b^Ty+1|\leq \delta e^T|y| \text{ is feasible}\}, \tag{4}$$

from which

$$r^s_\leq = \min_{y\geq 0,\ y\neq 0} \frac{\max\{\|A^Ty\|_\infty, |b^Ty+1|\}}{e^T|y|} = \min_{y\geq 0,\ y\neq 0} \frac{\|(y^TA, y^Tb+1)\|_\infty}{\|y\|_1}.$$

In view of Theorem 2, the value of $r^i_{\leq}$ can be expressed as

$$r^i_{\leq} = \inf\{\delta \geq 0;\ Ax - b \leq \delta E|x| + \delta e \text{ is feasible}\}, \tag{5}$$

from which

$$r^i_{\leq} = \min_{x \geq 0} \frac{\|(Ax-b)^+\|_\infty}{\|x\|_1 + 1}.$$

In view of Theorem 2 and Lemma 4, the value of $r^f_{\leq}$ can be expressed as

$$r^f_{\leq} = \inf\{\delta \geq 0;\ -A^T y \leq \delta E|y|,\ b^T y + 1 \leq \delta e^T|y| \text{ is feasible}\}, \tag{6}$$

from which

$$\begin{aligned} r^f_{\leq} &= \min_{y \geq 0,\ y \neq 0} \frac{\max\{0, \max_i(-A^T y)_i, b^T y + 1\}}{e^T|y|} \\ &= \min_{y \geq 0,\ y \neq 0} \frac{\|(-y^T A, y^T b + 1)^+\|_\infty}{\|y\|_1}. \quad \square \end{aligned}$$

Notice that the characterizations from the above Theorems 3 and 4 already appeared in the technical report [11], but without proofs. So this paper can be viewed as a publication of those results, among many others.

2.2. Properties

The following theorem shows some relations between the radii of solvability.

Theorem 5. *We have*

(i) $r^u_{=}$ *for* $Ax = b$ *is equal to* $r^u_{\leq}$ *for* $Ax \leq b,\ -Ax \leq -b$*,*
(ii) $r^i_{=}$ *for* $Ax = b$ *is equal to* $r^i_{\leq}$ *for* $Ax \leq b,\ -Ax \leq -b$*,*
(iii) $r^s_{=}$ *for* $Ax = b$ *is equal to* $r^f_{=}$ *for* $Ax^1 - Ax^2 = b$*,*
(iv) $r^s_{\leq}$ *for* $Ax \leq b$ *is equal to* $r^f_{\leq}$ *for* $Ax^1 - Ax^2 \leq b$*.*

Proof. In the following, we will use the fact that the solutions to systems $\boldsymbol{A}x = \boldsymbol{b}$ and $\boldsymbol{A}x \leq \boldsymbol{b}$, $-\boldsymbol{A}x \leq -\boldsymbol{b}$ are the same [12,13].

(i) We have

$$\begin{aligned} r^u_{=} &= \inf\{\delta \geq 0;\ [A - \delta E, A + \delta E]x = [b - \delta e, b + \delta e] \text{ is solvable}\} \\ &= \inf\{\delta \geq 0;\ [A - \delta E, A + \delta E]x \leq [b - \delta e, b + \delta e], \\ &\qquad - [A - \delta E, A + \delta E]x \leq -[b - \delta e, b + \delta e] \text{ is solvable}\} \\ &= r^u_{\leq}. \end{aligned}$$

(ii) We have

$$\begin{aligned} r^i_= &= \inf\{\delta \geq 0;\ [A - \delta E, A + \delta E]x = [b - \delta e, b + \delta e] \text{ is feasible}\} \\ &= \inf\{\delta \geq 0;\ [A - \delta E, A + \delta E]x \leq [b - \delta e, b + \delta e], \\ &\qquad\qquad - [A - \delta E, A + \delta E]x \leq -[b - \delta e, b + \delta e] \text{ is feasible}\} \\ &= r^i_\leq. \end{aligned}$$

(iii) We have by Lemmas 1 and 2

$$\begin{aligned} r^s_= &= \inf\{\delta \geq 0;\ [A - \delta E, A + \delta E]^T y = 0,\ [b - \delta e, b + \delta e]^T y \leq -1 \text{ is solvable}\} \\ &= \inf\{\delta \geq 0;\ [A - \delta E, A + \delta E]^T y \leq 0,\ -[A - \delta E, A + \delta E]^T y \leq 0, \\ &\qquad\qquad [b - \delta e, b + \delta e]^T y = -1 \text{ is solvable}\} \\ &= r^f_=. \end{aligned}$$

(iv) We have by Lemmas 3 and 4

$$\begin{aligned} r^s_\leq &= \inf\{\delta \geq 0;\ [A - \delta E, A + \delta E]^T y = 0,\ [b - \delta e, b + \delta e]^T y \leq -1 \text{ is feasible}\} \\ &= \inf\{\delta \geq 0;\ [A - \delta E, A + \delta E]^T y \leq 0,\ -[A - \delta E, A + \delta E]^T y \leq 0, \\ &\qquad\qquad [b - \delta e, b + \delta e]^T y = -1 \text{ is feasible}\} \\ &= r^f_\leq. \quad \square \end{aligned}$$

Comment. The above statements (i) and (ii) are by far not obvious. Their proof uses the fact that the solutions to systems $\boldsymbol{A}x = \boldsymbol{b}$ and $\boldsymbol{A}x \leq \boldsymbol{b}$, $-\boldsymbol{A}x \leq -\boldsymbol{b}$ are the same [12,13]. Similar transformations, however, needn't be equivalent since they cause the so called dependencies between interval parameters. This is also why we cannot state a simple analogy of these statements for the other radii of solvability and the statements (iii) and (iv) have different form.

Some cheap upper bounds for the radii are mentioned now.

Theorem 6. *We have*

(i) $r^u_= \leq r^i_= \leq \min_j\{\max_i |a_{ij}|, \max_i |b_i|\}$,
(ii) $r^f_= \leq r^s_= \leq \min_i \max_j |a_{ij}|$,
(iii) $r^u_\leq \leq r^i_\leq \leq \min_j\{\max_i (a_{ij})^+, \max_i(-b_i)^+\}$,
(iv) $r^f_\leq \leq r^s_\leq \leq \min_i \max_j\{|a_{ij}|, |b_i|\}$.

Proof. (i) If the minimum is equal to $\max_i |b_i|$, then put $b' := -b$ and the vector $x := 0$ solves $Ax = b + b'$. Otherwise, the minimum is equal to $\max_i |a_{ij}|$ for some j. In this case, put $b' := 0$ and $A' := (-A_{*j} + \varepsilon b)e_j^T$. Then the vector $x := \frac{1}{\varepsilon} e_j$ solves $(A + A')x = b + b'$ for every $\varepsilon > 0$.

(*ii*) Suppose that the minimum is attained at i and consider the ith equation $A_{i*}x = b_i$. Letting $A'_{i*} := -A_{i*}$, the equation $(A + A')_{i*}x = b_i$ reads $0^T x = b$, which is either unsolvable or becomes unsolvable after infinitesimal perturbation of the right-hand side.

(*iii*) If the minimum is equal to $\max_i(-b_i)^+$, then put $b' := (-b)^+$ and the vector $x := 0$ solves $Ax \le b + b'$. Otherwise, the minimum is equal to $\max_i(a_{ij})^+$ for some j. In this case, put $b' := 0$ and $A' := (-(A_{*j})^+ - \varepsilon e)e_j^T$. Then the vector $x := \frac{\max_i |b_i|}{\varepsilon} e_j$ solves $(A + A')x \le b + b'$ for every $\varepsilon > 0$.

(*iv*) Suppose that the minimum is attained at i and consider the ith inequality $A_{i*}x \le b_i$. Put $b'_i := -b_i - \varepsilon$ and $A'_{i*} := -A_{i*}$. Then the inequality $(A + A')_{i*}x \le (b + b')_i$ reads $0^T x \le -\varepsilon$, which is unsolvable for any $\varepsilon > 0$. □

Another basic properties are listed below. We denote by $r(A, b)$ the corresponding radius (of solvability, unsolvability, ...) for a system with the constraint matrix A and the right-hand side vector b.

Theorem 7. *We have*

(i) *For every* $\alpha \in \mathbb{R}$ *and* $r \in \{r^u_=, r^s_=, r^i_=, r^f_=\}$ *we have* $r(\alpha A, \alpha b) = |\alpha| r(A, b)$.
(ii) *For every* $\alpha \ge 0$ *and* $r \in \{r^u_\le, r^s_\le, r^i_\le, r^f_\le\}$ *we have* $r(\alpha A, \alpha b) = \alpha r(A, b)$.
(iii) *For every* $r \in \{r^u_=, r^i_=, r^u_\le, r^i_\le\}$ *we have* $r(\binom{A}{B}, \binom{b}{c}) \ge \max\{r(A, b), r(B, c)\}$.
(iv) *For every* $r \in \{r^s_=, r^f_=, r^s_\le, r^f_\le\}$ *we have* $r(\binom{A}{B}, \binom{b}{c}) \le \min\{r(A, b), r(B, c)\}$.

Proof. Obvious. □

In [3], it was shown that $r^u_=$ needn't be attained; that is, the corresponding system from the definition of $r^u_=$ is solvable for some A', b' such that $\|(A' \mid b')\|_{1,\infty} > r^u_=$, but for no A', b' such that $\|(A' \mid b')\|_{1,\infty} = r^u_=$. Similar properties have solvability-type radii. For instance, $r^s_= = 0$ for

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

but the zero is not attained.

On the other hand, $r^s_=$ is attained as long as A is nonsingular. In this case, the nearest singular matrix in $(1, \infty)$-norm is attained [4,5,14], and since the norm must be positive, the corresponding system is unsolvable for a suitable right-hand side.

2.3. Complexity

Theorem 8. *We have*

(i) *computing* $r^u_=$ *is an NP-hard problem,*
(ii) *computing* $r^s_=$ *is an NP-hard problem,*

(iii) *computing* $r^i_=$ *is a polynomial problem,*
(iv) *computing* $r^f_=$ *is an NP-hard problem.*

Proof. We will use the following NP-hard problem: Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. In [5,14], it was shown that determining the so called radius of nonsingularity

$$d(A) := \inf\{\|A'\|_{1,\infty};\ A + A' \text{ is singular}\} \tag{7}$$

is NP-hard.

(i) This case was already proved in [3], however, for the purpose of inapproximability results (Theorem 9) we show a different proof here.

Let $A \in \mathbb{R}^{n\times(n-1)}$, $b \in \mathbb{R}^n$, and suppose that $(A \mid b)$ is nonsingular. We claim that $d(A \mid b) = r^u_=$. If a perturbation $(\tilde{A} \mid \tilde{b})$ is nonsingular, then $\tilde{A}x = b'$ is unsolvable. If $(\tilde{A} \mid \tilde{b})$ is singular, then either $\tilde{b}$ is in the image of $\tilde{A}$ (meaning that $\tilde{A}x = \tilde{b}$ is solvable), or there is $y \in \mathbb{R}^{n-1}$, $y_k \neq 0$, such that $\tilde{A}y = 0$. In this case, $(\tilde{A} + \varepsilon\tilde{b}e_k^T)x = \tilde{b}$ is solvable with $\varepsilon > 0$ arbitrary; a solution is $x := \frac{1}{\varepsilon y_k}y$.

(ii) Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. We will prove that $d(A) = r^s_=$ for the system with the constraint matrix A and the right-hand side vector $b := 0$. If $(A + A')x = b'$ is unsolvable, then $A + A'$ must be singular. Conversely, if $A + A'$ is singular, then there is a vector b' not belonging to its image. Thus $(A + A')x = b'$ is unsolvable, where b' can be normalized such that $\|b'\|_\infty < \varepsilon$ for any $\varepsilon > 0$.

(iii) Due to (3), $r^i_=$ can be determined as

$$r^i_= = \inf\{\delta \geq 0;\ Ax - b \leq \delta Ex + \delta e,\ -Ax + b \leq \delta Ex + \delta e,\ x \geq 0\}.$$

This optimization problem has a form of a generalized linear fractional programming problem (GLFP), which is solvable in polynomial time using an interior point method [15,16].

(iv) By reduction from Theorem 8(ii) and using Theorem 5(iii). □

Theorem 9. *We have*

(i) *computing* $r^u_\leq$ *is an NP-hard problem,*
(ii) *computing* $r^s_\leq$ *is a polynomial problem,*
(iii) *computing* $r^i_\leq$ *is a polynomial problem,*
(iv) *computing* $r^f_\leq$ *is a polynomial problem.*

Proof. (i) By reduction from Theorem 8(i) and using Theorem 5(i).

(ii) Due to (4), $r^s_\leq$ can be determined as

$$\begin{aligned} r^s_\leq = \inf\{\delta \geq 0;\ & A^Ty \leq \delta Ey,\ -A^Ty \leq \delta Ey,\\ & b^Ty + 1 \leq \delta e^Ty,\ -b^Ty - 1 \leq \delta e^Ty,\ y \geq 0\}, \end{aligned}$$

which meets the form of GLFP.

(iii) By (5), $r^i_{\leq}$ can be expressed as

$$r^i_{\leq} = \inf\{\delta \geq 0;\ Ax - b \leq \delta Ex + \delta e,\ x \geq 0\},$$

which is again a GLFP.

(iv) Due to (6), $r^f_{\leq}$ can be determined as

$$r^f_{\leq} = \inf\{\delta \geq 0;\ -A^T y \leq \delta Ey,\ b^T y + 1 \leq \delta e^T y,\ y \geq 0\},$$

which is again a GLFP. □

The NP-hardness results can be even more strengthened to inapproximability statements.

Theorem 10. *Unless P = NP, there is no polynomial algorithm to compute*

(i) $r^u_{=}$ *with relative error* $\frac{1}{4}\min(m, n+1)^{-2}$,
(ii) $r^s_{=}$ *with relative error* $\frac{1}{4}\min(m, n)^{-2}$,
(iii) $r^f_{=}$ *with relative error* $\frac{1}{4}\min(m, \frac{1}{2}n)^{-2}$,
(iv) $r^s_{\leq}$ *with relative error* $\frac{1}{4}\min(\frac{1}{2}m, n+1)^{-2}$.

Proof. In [5], it was shown that computing the radius of nonsingularity (7) is NP-hard even with relative error $\frac{1}{4}n^{-2}$. The rest follows from the proofs of Theorems 8 and 9. □

On the other hand, the good news is that there is an effective algorithm provided a certain parameter of the problem is fixed.

Theorem 11. *Let* $A \in \mathbb{R}^{m\times n}$ *and* $b \in \mathbb{R}^m$. *There is a polynomial algorithm to compute*

(i) $r^u_{=}$ *and* $r^u_{\leq}$ *provided the number of variables (n) is fixed,*
(ii) $r^s_{=}$ *and* $r^f_{=}$ *provided the number of equations (m) is fixed.*

Proof. (i) As in the proof of Theorem 3, we have

$$r^u_{=} = \inf\{\delta \geq 0;\ |Ax - b| \leq \delta E|x| + \delta e \text{ is solvable}\}.$$

Denoting $s := \operatorname{sgn}(x) \in \{\pm 1\}^n$ the sign vector of x, we can write $|x| = \operatorname{diag}(s)x$, where $\operatorname{diag}(s)$ is the diagonal matrix with entries $s_1, \ldots, s_n$. Then $E|x| = E\operatorname{diag}(s)x = es^T x$, and we can express $r^u_{=}$ as

$$r^u_{=} = \inf_{s\in\{\pm 1\}^n} r_s,$$

where

$$r_s = \inf\{\delta \geq 0;\ Ax - b \leq \delta e s^T x + \delta e,\ -Ax + b \leq \delta e s^T x + \delta e,\ \operatorname{diag}(s)x \geq 0\}.$$

Thus, we reduced the problem of computing $r^u_=$ to solving 2^n problems of GLFP. This is a polynomial method as long as n is fixed.

Analogously, using

$$r^u_\leq = \inf\{\delta \geq 0;\ Ax - b \leq \delta E|x| + \delta e \text{ is solvable}\},$$

and decomposing the space $\mathbb{R}^n$ according to the signs of the entries of x, we can express $r^u_\leq$ in terms of solving 2^n problems of GLFP.

(ii) From

$$r^s_= = \inf\{\delta \geq 0;\ |A^T y| \leq \delta E|y|,\ |b^T y + 1| \leq \delta e^T |y| \text{ is solvable}\}$$

we can find $r^s_=$ by solving 2^m problems of GLFP (since $y \in \mathbb{R}^m$) and similarly for $r^f_=$ using

$$r^f_= = \inf\{\delta \geq 0;\ -A^T y \leq \delta E|y|,\ b^T y \leq \delta e^T |y| - 1 \text{ is solvable}\}. \quad \square$$

3. Extensions

3.1. General linear systems

Consider a general linear system in the form

$$Ax + By = b,\ Cx + Dy \leq d,\ x \geq 0, \tag{8}$$

comprising both equations and inequalities, and consider both the concepts of solvability and feasibility. Then the radius of unsolvability

$$r^u := \inf\left\{ \left\| \begin{pmatrix} A' & B' & b' \\ C' & D' & d' \end{pmatrix} \right\|_{1,\infty};\ (A + A')x + (B + B')y = b + b', \right.$$
$$\left. (C + C')x + (D + D')y \leq d + d',\ x \geq 0 \right\}$$

generalizes $r^u_=, r^i_=, r^u_\leq, r^i_\leq$, and the radius of solvability

$$r^s := \inf\left\{ \left\| \begin{pmatrix} A' & B' & b' \\ C' & D' & d' \end{pmatrix} \right\|_{1,\infty};\ (A + A')x + (B + B')y = b + b', \right.$$
$$\left. (C + C')x + (D + D')y \leq d + d',\ x \geq 0 \text{ is unsolvable} \right\}$$

generalizes $r^s_=, r^f_=, r^s_\leq, r^f_\leq$.

In order to characterize r^u, r^s we utilize the following description of the general interval linear system from [10].

Theorem 12. *The solution set of*

$$\boldsymbol{A}x + \boldsymbol{B}y = \boldsymbol{b},\ \boldsymbol{C}x + \boldsymbol{D}y \le \boldsymbol{d},\ x \ge 0, \tag{9}$$

is described by

$$|A^c x + B^c y - b^c| \le A^\Delta x + B^\Delta |y| + b^\Delta,$$
$$C^c x + D^c y - d^c \le C^\Delta x + D^\Delta |y| + d^\Delta,\ x \ge 0.$$

Theorem 13. *We have*

$$r^u = \min_{x\ge 0, y} \frac{\max\{\|Ax + By - b\|_\infty, \|(Cx + Dy - d)^+\|_\infty\}}{\|x\|_1 + \|y\|_1 + 1},$$
$$r^s = \min_{q\ge 0, p, (p,q)\ne(0,0)} \frac{\max\{\|(-A^T p - C^T q)^+\|_\infty, \|B^T p + D^T q\|_\infty, |b^T p + d^T q + 1|\}}{\|p\|_1 + \|q\|_1}.$$

Proof. In view of Theorem 12, the value of r^u can be expressed as

$$r^u = \inf\{\delta \ge 0;\, |Ax + By - b| \le \delta E x + \delta E|y| + \delta e,$$
$$Cx + Dy - d \le \delta E x + \delta E|y| + \delta e,\ x \ge 0\},$$

from which the formula for r^u follows.

By Farkas lemma (see the variant from [10]), the system (8) is not solvable iff and only if the system

$$A^T p + C^T q \ge 0,\ B^T p + D^T q = 0,\ b^T p + d^T q = -1,\ q \ge 0$$

is solvable. Hence by Theorem 12, the value of r^s can be expressed as

$$r^s = \inf\{\delta \ge 0;\, -A^T p - C^T q \le \delta E|p| + \delta E q,$$
$$|B^T p + D^T q| \le \delta E|p| + \delta E q,$$
$$|b^T p + d^T q + 1| \le \delta E|p| + \delta E q,\ q \ge 0\},$$

from which the formula for r^u follows. □

We can easily generalize the bounds from Theorem 6 as follows.

Theorem 14. *We have*

(i) $r^u \le \min_{j,\ell}\{\max_{i,k}\{|b_i|, (-d_k)^+|\}, \max_{i,k}\{|a_{ij}|, (c_{i\ell})^+\}, \max_{i,k}\{|b_{i\ell}|, (d_{k\ell})^+\}\}$,
(ii) $r^s \le \min_{i,k}\{\max_{j,\ell}\{|a_{ij}|, |b_{i\ell}|\}, \max_{j,\ell}\{|c_{kj}|, |d_{k\ell}|, |d_k|\}\}$.

Proof. (i) If the minimum is $\max_{i,k}\{|b_i|,(-d_k)^+|\}$, then $x=0$, $y=0$ solve the perturbed system with zero and non-negative right-hand side for equations and inequalities, respectively. If the minimum is $\max_{i,k}\{|a_{ij}|,(c_{i\ell})^+\}$, then $x=\varepsilon^{-1}e_j$, $y=0$ solve the perturbed system by $A':=-A+\varepsilon be_j^T$, $B':=0$, $b':=0$, $C':=(-C_{*j}^+ - \max_k |d_k|\varepsilon e)e_j^T$, $D':=0$, $d':=0$, where $\varepsilon>0$ is arbitrarily small. The third case is dealt with accordingly.

(ii) If the minimum is attained for an equation, then there is a perturbation leading to an unsolvable equation $0^Tx+0^Ty=q$ for some $q\neq 0$. If the minimum is attained for an inequality, then there is a perturbation leading to an unsolvable inequality $0^Tx+0^Ty\leq -\varepsilon$. □

3.2. Relative perturbations

Radii of solvability and unsolvability considered so far were defined as infimal $\delta\geq 0$ such that an independent perturbation of all system coefficients up to δ leads to unsolvable (resp. solvable) system. Now, we generalize this concept to more flexible perturbations.

Consider the general linear system (8). Given nonnegative matrices A_0, B_0, C_0, D_0 and vectors b_0, d_0, consider the interval linear system (9) with interval matrices and vectors

$$\boldsymbol{A}:=[A-\delta A_0, A+\delta A_0],\ \ \boldsymbol{B}:=[B-\delta B_0, B+\delta B_0],$$
$$\boldsymbol{C}:=[C-\delta C_0, C+\delta C_0],\ \ \boldsymbol{D}:=[D-\delta D_0, D+\delta D_0],$$
$$\boldsymbol{b}:=[b-\delta b_0, b+\delta b_0],\ \ \boldsymbol{d}:=[d-\delta d_0, d+\delta d_0].$$

For (8), define the radius of unsolvability and solvability respectively as

$$\rho^u:=\inf\{\delta\geq 0;\text{ the class (9) contains a solvable system}\},$$
$$\rho^s:=\inf\{\delta\geq 0;\text{ the class (9) contains an unsolvable system}\}.$$

Clearly, $\rho^u=r^u$ and $\rho^s=r^s$ if all radii matrices and vectors $A_0,\dots,d_0$ consists of ones. Another important case is $A_0=|A|,\dots,d_0=|d|$, where ρ^u and ρ^s correspond to maximal percentage perturbation of coefficients such that the system (8) remains unsolvable and solvable, respectively. In this case, it is easy to show that $\rho^u\leq 1$ since (9) will contain the system with zero coefficients. Analogously, $\rho^s\leq 1$ since (9) will contain the system with zero coefficients in the left-hand side and arbitrarily small negative coefficients in the right-hand side.

Theorem 15. *We have*

$$\rho^u=\min_{x\geq 0,y,(x,y)\neq(0,0)}\max\left\{\max_i\frac{|Ax+By-b|_i}{(A_0x+B_0|y|+b_0)_i},\ \max_j\frac{(Cx+Dy-d)_j}{(C_0x+D_0|y|+d_0)_j}\right\},$$

$$\rho^s = \min_{q \geq 0, p, (p,q) \neq (0,0)} \max \left\{ \max_i \frac{(-A^T p - C^T q)_i}{(A_0^T |p| + C_0^T q)_i}, \right.$$
$$\left. \max_j \frac{|B^T p + D^T q|_j}{(B_0^T |p| + D_0^T q)_j}, \frac{|b^T p + d^T q + 1|}{b_0^T |p| + d_0^T q} \right\},$$

where $\frac{0}{0} := 0$ *and* $\frac{a}{0} := \infty$, $a \neq 0$, *by convention.*

Proof. In view of Theorem 12, the value of ρ^u can be expressed as

$$\rho^u = \inf\{\delta \geq 0; |Ax + By - b| \leq \delta A_0 x + \delta B_0 |y| + \delta b_0,$$
$$Cx + Dy - d \leq \delta C_0 x + \delta D_0 |y| + \delta d_0,\ x \geq 0\},$$

from which the formula for ρ^u follows.

Analogously as in the proof of Theorem 13, the value of ρ^s can be expressed as

$$\rho^s = \inf\{\delta \geq 0;\ -A^T p - C^T q \leq \delta A_0^T |p| + \delta C_0^T q,$$
$$|B^T p + D^T q| \leq \delta B_0^T |p| + \delta D_0^T q,$$
$$|b^T p + d^T q + 1| \leq \delta b_0^T |p| + \delta d_0^T q,\ q \geq 0\},$$

from which the formula for ρ^u follows. $\square$

Remark 1. We could consider special cases of either equation, or inequalities, and distinguish (un)solvability and (in)feasibility as in Definition 1. Many properties from Section 2, however, are valid even in the context of more general perturbations considered here. For example, the polynomial cases from Theorems 8 and 9 remain valid since the problems reduce to GLFP.

As a generalization of Theorem 5, we can state that ρ^u will not change if we rewrite (9) as

$$\boldsymbol{A}x + \boldsymbol{B}y \leq \boldsymbol{b},\ \boldsymbol{A}x + \boldsymbol{B}y \geq \boldsymbol{b},\ \boldsymbol{C}x + \boldsymbol{D}y \leq \boldsymbol{d},\ x \geq 0,$$

where double appearances of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{b}$ are handled as independent interval objects. Analogously, ρ^s will not change if we rewrite (9) as

$$\boldsymbol{A}x + \boldsymbol{B}u - \boldsymbol{B}v = \boldsymbol{b},\ \boldsymbol{C}x + \boldsymbol{D}u - \boldsymbol{D}v \leq \boldsymbol{d},\ x, u, v \geq 0,$$

where double appearances of $\boldsymbol{B}$ and $\boldsymbol{D}$ are considered as independent.

4. Conclusion

We introduced the concept of radii of (un)solvability for linear systems of equations and inequalities. It can be seen as a generalization of regression in Chebyshev norm to inequality systems.

We characterized the radii by explicit formulae and discussed complexity questions, including inapproximability. Extensions to non-uniform perturbations were given, too.

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References

[1] I. Markovsky, S. Van Huffel, Overview of total least-squares methods, Signal Process. 87 (10) (2007) 2283–2302.
[2] S. Van Huffel, J. Vandewalle, The Total Least Squares Problem, SIAM, 1991.
[3] M. Hladík, M. Černý, Total least squares and Chebyshev norm, Proc. Comput. Sci. 51 (0) (2015) 1791–1800.
[4] D. Hartman, M. Hladík, Tight bounds on the radius of nonsingularity, in: M. Nehmeier, et al. (Eds.), Scientific Computing, Computer Arithmetic, and Validated Numerics, in: Lecture Notes in Computer Science, vol. 9553, ISBN 978-3-319-31769-4, 2016, in press.
[5] V. Kreinovich, A. Lakeyev, J. Rohn, P. Kahl, Computational Complexity and Feasibility of Data Processing and Interval Computations, Kluwer, Dordrecht, 1998.
[6] S.M. Rump, Bounds for the componentwise distance to the nearest singular matrix, SIAM J. Matrix Anal. Appl. 18 (1) (1997) 83–103.
[7] W. Oettli, W. Prager, Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides, Numer. Math. 6 (1964) 405–409.
[8] W. Gerlach, Zur Lösung linearer Ungleichungssysteme bei Störung der rechten Seite und der Koeffizientenmatrix, Math. Oper.forsch. Stat., Ser. Optim. 12 (1981) 41–43.
[9] M. Fiedler, J. Nedoma, J. Ramík, J. Rohn, K. Zimmermann, Linear Optimization Problems with Inexact Data, Springer, New York, 2006.
[10] M. Hladík, Weak and strong solvability of interval linear systems of equations and inequalities, Linear Algebra Appl. 438 (11) (2013) 4156–4165.
[11] J. Rohn, A Manual of Results on Interval Linear Problems, Technical Report 1164, Institute of Computer Science, Academy of Sciences, Czech Republic, Prague, http://www.library.sk/arl-cav/en/detail/?&idx=cav_un_epca*0381706, 2012.
[12] W. Li, A note on dependency between interval linear systems, Optim. Lett. 9 (4) (2015) 795–797.
[13] J. Rohn, Miscellaneous Results on Linear Interval Systems, Freiburger Intervall-Berichte, vol. 85/9, Albert-Ludwigs-Universität, Freiburg, 1985.
[14] S. Poljak, J. Rohn, Checking robust nonsingularity is NP-hard, Math. Control Signals Systems 6 (1) (1993) 1–9.
[15] R.W. Freund, F. Jarre, An interior-point method for multifractional programs with convex constraints, J. Optim. Theory Appl. 85 (1) (1995) 125–161.
[16] Y.E. Nesterov, A.S. Nemirovskii, An interior-point method for generalized linear-fractional programming, Math. Program. 69 (1B) (1995) 177–204.