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Regularity of Interval Matrices and Theorems of the Alternatives

Dedicated to Professor Miroslav Fiedler on the occasion of his 80th birthday

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Abstract. We give several characterizations of regularity of interval matrices. All of them have to do with solvability of certain systems of nonlinear equations or inequalities. The most illustrative of them is the following one: an interval matrix $[A_c - \Delta, A_c + \Delta]$ is regular if and only if the nonlinear inequality $|x| > \Delta|A_c^{-1}x|$ has a solution in each orthant. These results are then applied to derive two theorems of the alternatives for inequalities with absolute values.

1. Introduction

In this paper we use the following notations. Matrix inequalities, as $A \leq B$ or $A < B$, are understood componentwise. The absolute value of a matrix $A = (a_{ij})$ is defined by $A = (|a_{ij}|)$. The same notations also apply to vectors that are considered one-column matrices. I is the unit matrix, e_j is the jth column of I, and $e = (1, \dots, 1)^T$ is the vector of all ones. $Y_n = \{y \mid |y| = e\}$ is the set of all ±1-vectors in $\mathbb{R}^n$, so that its cardinality is 2^n. For each $y \in Y_n$ we denote

$$T_y = \operatorname{diag}(y_1, \dots, y_n) = \begin{pmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_n \end{pmatrix},$$

and $\mathbb{R}^n_y = \{x \mid T_y x \geq 0\}$ is the orthant prescribed by the ±1-vector $y \in Y_n$.

As is well known, a square interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta] = \{A \mid A_c - \Delta \leq A \leq A_c + \Delta\}$ is called regular if each $A \in \mathbf{A}$ is nonsingular. More than ten necessary and sufficient regularity conditions are given in Theorem 5.1 in [6]. All of them are finitely verifiable, but all of them exhibit exponential complexity. This fact was explained in late 1980's by Poljak and Rohn [2] who proved that checking regularity of interval matrices is a co-NP-complete problem, so that necessary and sufficient regularity conditions verifiable in polynomial time cannot be expected to exist provided the famous conjecture $P \neq NP$ is true.

In this paper we give two other sets of necessary and sufficient regularity conditions. All of them have to do with solvability of some systems of nonlinear equations or inequalities. The most illustrative among them is the following one: $[A_c - \Delta, A_c + \Delta]$ is regular if and only if the nonlinear equation

$$|x| > \Delta|A_c^{-1}x|$$

has a solution in each orthant (Theorem 3.2, (v)). None of these conditions can escape the exponential complexity (there are 2^n orthants in $\mathbb{R}^n$), but their formulations are "more elegant" since exponentiality is somewhat hidden there.

The layout of the paper is as follows. In Section 2 we formulate an auxiliary existence lemma which, although little known, is very useful in interval contexts. It is used in Section 3 at a key point of the main proof. The results are formulated in two theorems. In Theorem 3.1 nonsingularity of A_c is not assumed, but as a result the formulations are a little burdensome. Assuming nonsingularity of A_c in Theorem 3.2, we obtain more elegant formulations of the type "... [unique] solvability in each orthant." As a direct consequence we then prove in Section 4 two theorems of the alternatives for inequalities involving absolute values.

2. An Auxiliary Existence Lemma

At a key point of the main proof in Section 3 we shall employ the following existence lemma. It is formulated here in its full generality for rectangular linear systems, although we shall utilize it for the square case only.

LEMMA 2.1. *Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and let for each $y \in Y_m$ the inequality*

$$T_yAx \geq T_yb \tag{2.1}$$

have a solution x_y. Then the equation

$$Ax = b \tag{2.2}$$

has a solution in the set Conv $\{x_y \mid y \in Y_m\}$.

Here, Conv denotes the convex hull. Two proofs of this lemma are available: a shorter existence proof in [5], which makes use of the Farkas lemma, and a lengthy constructive proof in [4], where it is shown that a solution x of (2.2) can be constructed from the solutions x_y of (2.1) by means of a certain finite "halve and delete" algorithm.

3. Regularity Conditions

First we shall not assume nonsingularity of the midpoint matrix A_c. Then we have the following necessary and sufficient regularity conditions:

THEOREM 3.1. *For a square interval matrix* $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$, *the following assertions are equivalent:*

(i) **A** *is regular,*

(ii) for each vector $a > 0$ *the equation*

$$|A_c x| = \Delta|x| + a \tag{3.1}$$

has for each $y \in Y_n$ *a unique solution satisfying* $A_c x \in \mathbb{R}^n_y$*,*

(iii) the equation

$$|A_c x| = \Delta|x| + e$$

has for each $y \in Y_n$ *a unique solution satisfying* $A_c x \in \mathbb{R}^n_y$*,*

(iv) there exists an $a > 0$ *such that the equation (3.1) has for each* $y \in Y_n$ *a solution satisfying* $A_c x \in \mathbb{R}^n_y$*,*

(v) for each $y \in Y_n$ *the inequality*

$$|A_c x| > \Delta|x| \tag{3.2}$$

has a solution satisfying $A_c x \in \mathbb{R}^n_y$*.*

Proof. We shall prove that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i)$ hold. The implications $(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$ are straightforward, so that we are left with proving $(i) \Rightarrow (ii)$ and $(v) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$: In [6, Theorem 2.2], it is stated that if **A** is regular, then for each $y \in Y_n$ and each right-hand side interval vector $\mathbf{b} = [b_c - \delta, b_c + \delta]$ the nonlinear equation

$$A_c x - T_y \Delta|x| = b_c + T_y \delta$$

has a unique solution. Let us apply the result to the right-hand side $\mathbf{b} = [-a, a]$. Then the theorem says that for each $y \in Y_n$ the equation

$$A_c x - T_y \Delta|x| = T_y a$$

has a unique solution, which then satisfies $T_y A_c x = \Delta|x| + a \geq a > 0$, hence $T_y A_c x > 0$ and thus also $A_c x \in \mathbb{R}^n_y$ and $|A_c x| = \Delta|x| + a$. If x' is another solution of (3.1) with $T_y A_c x' \in \mathbb{R}^n_y$, then it satisfies $A_c x' - T_y \Delta|x'| = T_y a$ and thus $x' = x$ in view of the uniqueness of the solution of this equation stated in the above-quoted theorem.

$(v) \Rightarrow (i)$: Let for each $y \in Y_n$ the inequality (3.2) have a solution x_y satisfying $T_y A_c x_y \geq 0$. Then $T_y A_c x_y - \Delta|x_y| > 0$, so that there exists a positive real number α_y such that

$$\alpha_y (T_y A_c x_y - \Delta|x_y|) \geq e. \tag{3.3}$$

Now, take an $A \in \mathbf{A}$ and $j \in \{1, \ldots, n\}$. We shall prove that the equation $Ax = e_j$ has a solution. To this end, take an arbitrary $y \in Y_n$. Since $|T_y(A - A_c)\alpha_y x_y| \leq \Delta \alpha_y |x_y|$, we have $T_y(A - A_c)\alpha_y x_y \geq -\Delta\alpha_y|x_y|$ and hence

$$\begin{aligned} T_y(A\alpha_y x_y - e_j) &= T_y A_c \alpha_y x_y + T_y(A - A_c)\alpha_y x_y - T_y e_j \\ &\geq T_y A_c \alpha_y x_y - \Delta\alpha_y|x_y| - y_j e_j \\ &\geq e - y_j e_j \geq 0 \end{aligned}$$

because of (3.3) and of the fact that $y_j = \pm 1$. We have proved that for given $A \in \mathbf{A}$ and $j \in \{1, \dots, n\}$ the inequality

$$T_y A x \geq T_y e_j$$

has a solution for each $y \in Y_n$ (namely, $x = \alpha_y x_y$). This, according to Lemma 2.1, means that the equation

$$Ax = e_j$$

has a solution, say $x^{(j)}$. Here, $j \in \{1, \dots, n\}$ was arbitrary. Hence, if we construct the matrix $X = (x^{(1)}, \dots, x^{(n)})$, then it satisfies $AX = I$, which means that A is invertible and thus nonsingular, hence $\mathbf{A}$ is regular. This concludes the proof of the implication $(v) \Rightarrow (i)$, and thus also of the whole theorem. □

Now, the formulation of Theorem 3.1 essentially simplifies if we assume that the midpoint matrix A_c is nonsingular. Then we have:

THEOREM 3.2. *For a square interval matrix* $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ *with nonsingular* A_c*, the following assertions are equivalent:*

(i) $\mathbf{A}$ *is regular,*

(ii) for each vector $a > 0$ *the equation*

$$|x| = \Delta|A_c^{-1}x| + a \tag{3.4}$$

has in each orthant a unique solution,

(iii) the equation

$$|x| = \Delta|A_c^{-1}x| + e$$

has in each orthant a unique solution,

(iv) there exists an $a > 0$ *such that the equation (3.4) has in each orthant a solution,*

(v) the inequality

$$|x| > \Delta|A_c^{-1}x| \tag{3.5}$$

has in each orthant a solution.

Proof. If $\mathbf{A}$ is regular, then according to Theorem 3.1, (*ii*), for each $a > 0$ and $y \in Y_n$ the equation

$$|A_c x| = \Delta|x| + a$$

has a unique solution x_y satisfying $A_c x_y \in \mathbb{R}^n_y$. Put $\tilde{x}_y = A_c x_y$, then for each $y \in Y_n$ the equation

$$|x| = \Delta|A_c^{-1}x| + a \tag{3.6}$$

has a unique solution $\tilde{x}_y$ satisfying $\tilde{x}_y \in \mathbb{R}^n_y$. This proves that the equation (3.6) has a unique solution in each orthant. Thereby we have proved the implication (*i*) $\Rightarrow$ (*ii*). The implications (*ii*) $\Rightarrow$ (*iii*) $\Rightarrow$ (*iv*) $\Rightarrow$ (*v*) are again obvious. If the inequality $|x| > \Delta|A_c^{-1}x|$ has a solution in each orthant, then for each $y \in Y_n$ it has a solution $\tilde{x}_y$ satisfying $\tilde{x}_y \in \mathbb{R}^n_y$. Put $x_y = A_c^{-1}\tilde{x}_y$, then $|A_c x_y| > \Delta|x_y|$ and $A_c x_y \in \mathbb{R}^n_y$, hence by the assertion (*v*) of Theorem 3.1 the interval matrix $\mathbf{A}$ is regular. This proves (*v*) $\Rightarrow$ (*i*), and thereby also the mutual equivalence of all five assertions. □

Finally we shall show that the explicit use of A_c^{-1} in (3.5) can be avoided while still keeping the property of existence of a solution in each orthant.

THEOREM 3.3. *An* $n \times n$ *interval matrix* $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ *is regular if and only if there exists a matrix* $R \in \mathbb{R}^{n\times n}$ *such that the inequality*

$$|x| > |(I - A_c R)x| + \Delta|Rx| \tag{3.7}$$

has a solution in each orthant.

Proof. If $\mathbf{A}$ is regular, then by Theorem 3.2, (*v*), the inequality (3.5) has a solution in each orthant; this is just the inequality (3.7) for $R = A_c^{-1}$. Conversely, let for each $y \in Y_n$ the inequality (3.7) have a solution $\tilde{x}_y \in \mathbb{R}^n_y$. Then from (3.7) it follows

$$\begin{aligned} \Delta|R\tilde{x}_y| &< |\tilde{x}_y| - |(I - A_c R)\tilde{x}_y| = \Big||\tilde{x}_y| - |(I - A_c R)\tilde{x}_y|\Big| \\ &\le |\tilde{x}_y - (\tilde{x}_y - A_c R\tilde{x}_y)| = |A_c R\tilde{x}_y|, \end{aligned}$$

hence $x_y := R\tilde{x}_y$ satisfies

$$|A_c x_y| > \Delta|x_y|.$$

Next, from (3.7) we have

$$|\tilde{x}_y - A_c x_y| = |(I - A_c R)\tilde{x}_y| < |\tilde{x}_y|,$$

which shows that $A_c x_y$ belongs to the same orthant as $\tilde{x}_y$, hence $A_c x_y \in \mathbb{R}^n_y$. We have proved that for each $y \in Y_n$ the inequality $|A_c x| > \Delta|x|$ has a solution (namely, x_y) satisfying $A_c x \in \mathbb{R}^n_y$. By Theorem 3.1, (*v*), this implies regularity of $\mathbf{A}$. □

4. Theorems of the Alternatives

As direct consequences of the previous results we obtain two nontrivial theorems of the alternatives for inequalities involving absolute values.

THEOREM 4.1. *For each* $A, B \in \mathbb{R}^{n\times n}$, *exactly one of the following two alternatives holds:*

(a) for each $y \in Y_n$ the inequality

$$|Ax| > |B||x|$$

has a solution x satisfying $Ax \in \mathbb{R}^n_y$,

(b) the inequality

$$|Ax| \leq |B||x|$$

has a nontrivial solution.

Proof. Consider the interval matrix $\mathbf{A} = [A - |B|, A + |B|]$. By assertion (v) of Theorem 3.1, regularity of $\mathbf{A}$ is equivalent to (a), and by the Oettli-Prager theorem [1] (applied to the interval linear system $[A - |B|, A + |B|]x = [0, 0]$) singularity of $\mathbf{A}$ is equivalent to (b). This proves that exactly one of the alternatives (a), (b) holds. □

As before, we get a more smooth formulation of the first alternative if we assume nonsingularity of A.

THEOREM 4.2. *Let $A, B \in \mathbb{R}^{n\times n}$, A nonsingular. Then exactly one of the following two alternatives holds:*

(a) the inequality

$$|x| > |B||Ax|$$

has a solution in each orthant,

(b) the inequality

$$|x| \leq |B||Ax|$$

has a nontrivial solution.

Proof. The proof goes along the same lines as the proof of Theorem 4.1 when employing the interval matrix $\mathbf{A} = [A^{-1} - |B|, A^{-1} + |B|]$ and the assertion (v) of Theorem 3.2. □

We have two corollaries. The first one is obtained directly from Theorem 4.1.

COROLLARY 4.1. *For each $A, B \in \mathbb{R}^{n\times n}$, at least one of the inequalities*

$$\begin{aligned} |Ax| &> |B||x|, \\ |Ax| &\leq |B||x| \end{aligned}$$

has a nontrivial solution.

The second corollary is obtained from Theorem 4.2 by setting $B = I$.

COROLLARY 4.2. *For a nonsingular matrix $A \in \mathbb{R}^{n\times n}$, exactly one of the following two alternatives holds:*

(a) the inequality

$$|x| > |Ax|$$

has a solution in each orthant,

(b) the inequality

$$|x| \leq |Ax|$$

has a nontrivial solution.

We note that another related theorem of the alternatives concerning solvability of the equation $Ax + B|x| = b$ was given in [3].

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