Institute of Computer Science
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A Bendixson-Type Theorem for Eigenvalues of Interval Matrices

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Technical report No. V-1184

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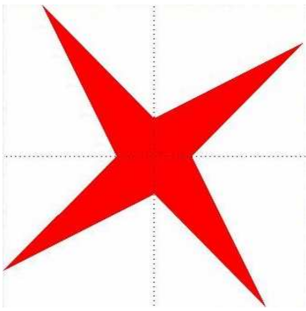
Abstract:

We describe a rectangle in complex plane which encloses all eigenvalues of an interval matrix. Special cases of symmetric and skew-symmetric interval matrices are also considered.[1]



Keywords:
Interval matrix, eigenvalues, enclosure, symmetric interval matrix, skew-symmetric interval matrix.

[1]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

1 Introduction

In this report we describe a rectangle in complex plane enclosing all eigenvalues of an interval matrix (Theorem 1 and Corollary 2). Special cases of a symmetric or skew-symmetric interval matrix are handled in Corollaries 3 and 4. These results are obtained as simplifications of Theorem 2 in [2].

2 The results

Our main result is formulated as follows.

Theorem 1. *Let* $\boldsymbol{A} = [A_c - \Delta, A_c + \Delta]$ *be a square interval matrix. Then for each eigenvalue* λ *of each* $A \in \boldsymbol{A}$ *we have*

$$\lambda_{\min}(A'_c) - \varrho(\Delta') \leq \operatorname{Re}\lambda \leq \lambda_{\max}(A'_c) + \varrho(\Delta'), \tag{2.1}$$

$$-\sigma_{\max}(A''_c) - \varrho(\Delta') \leq \operatorname{Im}\lambda \leq \sigma_{\max}(A''_c) + \varrho(\Delta'), \tag{2.2}$$

where

$$\begin{aligned} A'_c &= \tfrac{1}{2}(A_c + A_c^T), \\ A''_c &= \tfrac{1}{2}(A_c - A_c^T), \\ \Delta' &= \tfrac{1}{2}(\Delta + \Delta^T). \end{aligned}$$

Proof. In [2, Thm. 2] it is proved that under the current assumptions and notation there holds

$$\lambda_{\min}(A'_c) - \varrho(\Delta') \leq \operatorname{Re}\lambda \leq \lambda_{\max}(A'_c) + \varrho(\Delta'),$$

$$\lambda_{\min}(A'''_c) - \varrho(\Delta'') \leq \operatorname{Im}\lambda \leq \lambda_{\max}(A'''_c) + \varrho(\Delta''),$$

where

$$\begin{aligned} A'_c &= \tfrac{1}{2}(A_c + A_c^T), \\ \Delta' &= \tfrac{1}{2}(\Delta + \Delta^T), \\ A'''_c &= \begin{pmatrix} 0 & A''_c \\ A''^T_c & 0 \end{pmatrix}, \\ \Delta'' &= \begin{pmatrix} 0 & \Delta' \\ \Delta'^T & 0 \end{pmatrix}. \end{aligned}$$

This proves (2.1). To prove (2.2), we shall use the Jordan-Wielandt theorem [3, Thm. 4.2] according to which a matrix

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

(with $B \in \mathbb{R}^{n\times n}$) has eigenvalues $\sigma_1(B) \geq \cdots \geq \sigma_n(B) \geq -\sigma_n(B) \geq \ldots \geq -\sigma_1(B)$. Thus $\lambda_{\max}(A'''_c) = \sigma_{\max}(A''_c)$, $\lambda_{\min}(A'''_c) = -\sigma_{\max}(A''_c)$, and $\varrho(\Delta'') = \sigma_{\max}(\Delta') = \varrho(\Delta')$ (because Δ' is symmetric and nonnegative), whereby we are done. □

For other formulations, let us introduce the following notation for the set of all eigenvalues:

$$\Lambda(\boldsymbol{A}) = \{ \lambda \in \mathbb{C} \mid Ax = \lambda x,\ x \in \mathbb{C}^n,\ x \neq 0,\ A \in \boldsymbol{A} \}.$$

Notice that $\Lambda(\boldsymbol{A})$ is symmetric with respect to the real axis because, as well known, each $A \in \boldsymbol{A}$ together with an eigenvalue $\lambda = a + bi$ also possesses the eigenvalue $\overline{\lambda} = a - bi$.

Corollary 2. *For each square interval matrix* $\boldsymbol{A} = [A_c - \Delta, A_c + \Delta]$ *there holds*

$$\Lambda(\boldsymbol{A}) \subseteq [\lambda_{\min}(A'_c) - \varrho(\Delta'),\ \lambda_{\max}(A'_c) + \varrho(\Delta')] \times [-\sigma_{\max}(A''_c) - \varrho(\Delta'),\ \sigma_{\max}(A''_c) + \varrho(\Delta')],$$

where A'_c, A''_c *and* Δ' *are as in Theorem 1.*

This is merely a reformulation of Theorem 1 expressing the set of all eigenvalues of $\boldsymbol{A}$ as a subset of a rectangle in complex plane. This rectangle is also symmetric with respect to real axis.

For an interval matrix $\boldsymbol{A}$, its transpose is defined by

$$\boldsymbol{A}^T = \{ A^T \mid A \in \boldsymbol{A} \}.$$

A square interval matrix $\boldsymbol{A}$ is called symmetric if $\boldsymbol{A}^T = \boldsymbol{A}$, and it is said to be skew-symmetric if $\boldsymbol{A}^T = -\boldsymbol{A}$. It is not difficult to prove that $\boldsymbol{A} = [A_c - \Delta, A_c + \Delta]$ is symmetric if and only if both A_c and Δ are symmetric, and that it is skew-symmetric if and only if A_c is skew-symmetric and Δ is symmetric.

The next two corollaries describe enclosures of sets of eigenvalues of symmetric and skew-symmetric interval matrices, respectively.

Corollary 3. *For a symmetric interval matrix* $\boldsymbol{A} = [A_c - \Delta, A_c + \Delta]$ *we have*

$$\Lambda(\boldsymbol{A}) \subseteq [\lambda_{\min}(A_c) - \varrho(\Delta),\ \lambda_{\max}(A_c) + \varrho(\Delta)] \times [-\varrho(\Delta),\ \varrho(\Delta)].$$

Proof. Obviously, in this case $A'_c = A_c$, $A''_c = 0$ and $\Delta' = \Delta$, from which the result follows. □

Corollary 4. *For a skew-symmetric interval matrix* $\boldsymbol{A} = [A_c - \Delta, A_c + \Delta]$ *we have*

$$\Lambda(\boldsymbol{A}) \subseteq [-\varrho(\Delta),\ \varrho(\Delta)] \times [-\sigma_{\max}(A_c) - \varrho(\Delta),\ \sigma_{\max}(A_c) + \varrho(\Delta)].$$

Proof. This is a consequence of the facts that $A'_c = 0$, $A''_c = A_c$, and $\Delta' = \Delta$. □

Notice that in the latter case the enclosure of the eigenvalue set is symmetric with respect to the origin.

3 Acknowledgement

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