Institute of Computer Science
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A Class of Explicitly Solvable Absolute Value Equations

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Technical report No. V-1202

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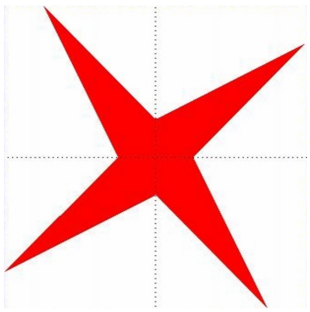
Abstract:

We describe explicit formulae for the solution of a special case of absolute value equations.[2]



Keywords:
Absolute value equation, solution, explicit form.

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[2]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

1 Introduction

Since its introduction in [3] ten years ago, the absolute value equation

$$Ax + B|x| = b$$

($A, B \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$) has received quite a bit of attention. Various authors have proposed numerous methods for finding its solution (as e.g. via linear programming, nonlinear programming, etc.), but it seems that nobody has attempted so far to find an explicit form of the‘ solution in some special cases. In this report we bring some result in this direction (Theorem 1). The assumption imposed on the right-hand side b may seem to be too restrictive, but even this partial result makes it possible to carry out the proof of the Hansen-Bliek-Rohn optimality result [2] in a new, more convincing fashion, which is possibly going to appear elsewhere.

Notation used: I is the identity matrix, e_k stands for the kth column of I, and the maximum of two vectors is taken entrywise.

2 The result

Our main result is formulated as follows.

Theorem 1. *Let $\Delta \geq 0$, let the matrix $M = (I - \Delta)^{-1}$ satisfy*

$$M \geq I, \tag{2.1}$$

and let there exist a k such that $b_i \geq 0$ for each $i \neq k$. Then the equation

$$x - \Delta|x| = b \tag{2.2}$$

has a unique solution

$$x = \max\{Mb,\ Mb - \tfrac{2(Mb)_k}{2M_{kk}-1}(M - I)e_k\}. \tag{2.3}$$

Moreover,

$$|x| = \max\{Mb,\ Mb - \tfrac{2(Mb)_k}{2M_{kk}-1}Me_k\}. \tag{2.4}$$

Proof. In view of nonnegativity of Δ, the assumption (2.1) implies that $\varrho(\Delta) < 1$, hence $\Delta^j \to 0$ as $j \to \infty$, so that the sequence $\{x_i\}_{i=0}^{\infty}$ defined by $x_0 = 0$ and

$$x_{i+1} = \Delta|x_i| + b \qquad (i = 0, 1, \ldots) \tag{2.5}$$

is Cauchian, thus convergent, $x_i \to x$. Taking the limit for $i \to \infty$ in (2.5), we obtain

$$x = \Delta|x| + b, \tag{2.6}$$

hence x is a solution of (2.2). Assume that x' also solves (2.2). Then from

$$x - x' = \Delta(|x| - |x'|)$$

we obtain

$$|x - x'| \leq \Delta|x - x'|$$

and

$$(I - \Delta)|x - x'| \leq 0,$$

and premultiplying this inequality by the nonnegative matrix M leads to conclusion that

$$|x - x'| \leq 0,$$

hence $x = x'$ and the solution x of (2.2) is unique.

Now, the assumption of nonnegativity of b_i for each $i \neq k$ implies in the light of (2.6) that $x_i \geq 0$ for $i \neq k$, so that $|x_i| = x_i$ for each $i \neq k$ and consequently we can write

$$|x| = x + (|x_k| - x_k)e_k. \tag{2.7}$$

Substituting this expression into (2.6), we obtain

$$(I - \Delta)x = (|x_k| - x_k)\Delta e_k + b$$

and by premultiplying by M we get

$$x = (|x_k| - x_k)(M - I)e_k + Mb \tag{2.8}$$

since $M = (I - \Delta)^{-1}$ implies $M\Delta = M - I$. From this we can see that if $x_k \geq 0$, then

$$x = Mb.$$

If $x_k < 0$, then $|x_k| = -x_k$ and from the kth equation in (2.8)

$$x_k = (|x_k| - x_k)(M_{kk} - 1) + (Mb)_k \tag{2.9}$$

we have

$$x_k = \frac{(Mb)_k}{2M_{kk} - 1} \tag{2.10}$$

(since $M_{kk} \geq 1$ implies $2M_{kk} - 1 \geq 1$), and finally from (2.8)

$$x = Mb - \tfrac{2(Mb)_k}{2M_{kk}-1}(M - I)e_k.$$

Notice now that $2M_{kk} - 1$ is positive and $(M - I)e_k$ is nonnegative. Thus, if $x_k \geq 0$, then $(Mb)_k \geq 0$ and

$$x = Mb \geq Mb - \tfrac{2(Mb)_k}{2M_{kk}-1}(M - I)e_k,$$

hence

$$x = \max\{Mb,\, Mb - \tfrac{2(Mb)_k}{2M_{kk}-1}(M - I)e_k\}, \tag{2.11}$$

and if $x_k < 0$, then $(Mb)_k < 0$ (for otherwise (2.9) would imply $x_k \geq 0$), hence

$$x = Mb - \tfrac{2(Mb)_k}{2M_{kk}-1}(M - I)e_k \geq Mb,$$

so that (2.11) again holds. This proves (2.3).

To prove (2.4), we proceed in a similar way. If $x_k \geq 0$, then $(Mb)_k \geq 0$ and

$$|x| = x = Mb \geq Mb - \tfrac{2(Mb)_k}{2M_{kk}-1} Me_k,$$

hence

$$|x| = \max\{Mb,\, Mb - \tfrac{2(Mb)_k}{2M_{kk}-1} Me_k\}, \tag{2.12}$$

whereas if $x_k < 0$, then $(Mb)_k < 0$ and from (2.7), (2.10) we have

$$|x| = Mb - \tfrac{2(Mb)_k}{2M_{kk}-1}(M-I)e_k - \tfrac{2(Mb)_k}{2M_{kk}-1} e_k = Mb - \tfrac{2(Mb)_k}{2M_{kk}-1} Me_k \geq Mb,$$

which is again (2.12). This proves (2.4), and the proof is complete. □

We can also formulate the result in the following more explicit form.

Theorem 2. *Under assumptions and notation of Theorem 1, the unique solution x of the equation (2.2) is given by*

$$x = \begin{cases} Mb & \text{if } (Mb)_k \geq 0, \\ Mb - \tfrac{2(Mb)_k}{2M_{kk}-1}(M-I)e_k & \text{if } (Mb)_k < 0, \end{cases}$$

and its absolute value by

$$|x| = \begin{cases} Mb & \text{if } (Mb)_k \geq 0, \\ Mb - \tfrac{2(Mb)_k}{2M_{kk}-1} Me_k & \text{if } (Mb)_k < 0. \end{cases}$$

Proof. This is an immediate consequence of (2.3), (2.4). □

3 Application

With the help of Theorem 1 it is possible to derive closed-form formulae for matrices Q_z (introduced in [4]) for interval matrices of the form $[I - \Delta, I + \Delta]$, which then leads to a new proof of the Hansen-Bliek-Rohn optimality result, as mentioned in the Introduction.

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