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Explicit Form of Matrices Q_z for an Interval Matrix with Unit Midpoint

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Technical report No. V-1206

20.02.2014

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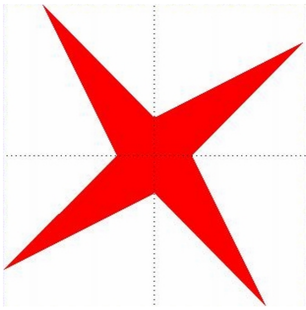
Abstract:

We give three versions of explicit formulae for matrices Q_z for an interval matrix with unit midpoint.[2]



Keywords:
Interval matrix, Q_z matrix, unit midpoint, explicit formula.

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[2]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

1 Introduction

Matrices Q_z, first appearing in [5], may be best introduced by way of the following theorem.

Theorem 1. *If an $n \times n$ interval matrix $\boldsymbol{A} = [A_c - \Delta, A_c + \Delta]$ is regular,*[3] *then for each $z \in \{-1, 1\}^n$ the nonlinear matrix equation*

$$QA_c - |Q|\Delta T_z = I \tag{1.1}$$

has a unique matrix solution Q_z.

Comment. Here the absolute value of a matrix is understood entrywise, I is the identity matrix and $T_z = \mathrm{diag}(z)$ denotes the diagonal matrix with diagonal vector z.

Proof. Because $\boldsymbol{A}$ is regular, its transpose $\boldsymbol{A}^T = \{\, A^T \mid A \in \boldsymbol{A} \,\} = [A_c^T - \Delta^T, A_c^T + \Delta^T]$ is also regular, hence by [3, Thm. 5.1, Assertion (A3)] for each $z \in \{-1, 1\}^n$ the equation

$$A_c^T B - T_z \Delta^T |B| = I \tag{1.2}$$

has a unique matrix solution B_z. Then

$$B_z^T A_c - |B_z^T| \Delta T_z = I,$$

hence $Q_z = B_z^T$ solves (1.1), and its uniqueness follows from that of (1.2). $\square$

A general algorithm **qzmatrix** for computing Q_z based on the algorithm **absvaleqn** for solving absolute value equations was described in [5]. Neither this nor any other known result gives any clue about the shape of these matrices. In this report we describe an explicit form of matrices Q_z for interval matrices with unit midpoint, i.e., satisfying $A_c = I$. That is, we look for explicit form of the solution of the equation

$$Q - |Q|\Delta T_z = I \tag{1.3}$$

where Δ is an arbitrary nonnegative matrix bound only by regularity requirement (see below). In this way we make a step towards our main goal, namely a new proof of the Hansen-Bliek-Rohn optimality result [4].

2 Matrices Q_z

For a nonnegative square matrix Δ put

$$M = (I - \Delta)^{-1}.$$

It is known that the following four assertions are equivalent:

(i) $[I - \Delta, I + \Delta]$ is regular,
(ii) $M \geq I$,
(iii) $M \geq 0$,
(iv) $\varrho(\Delta) < 1$

[3] I.e., each $A \in \boldsymbol{A}$ is nonsingular.

(see [6], [2]). Thus any of (ii)-(iv) can be used as a regularity condition. We choose (ii) because we need the fact that each diagonal entry of M is greater or equal than one. Now we have the following explicit description of solution of the equation (1.3).

Theorem 2. *Let $M \geq I$. Then for each $z \in \{-1,1\}^n$ the matrix Q_z is given rowwise by*

$$(Q_z)_{k\bullet} = \begin{cases} M_{k\bullet}T_z & \text{if } z_k = 1, \\ ((\mu_k - 1)M_{k\bullet} - \mu_k e_k^T)T_z & \text{if } z_k = -1, \end{cases} \tag{2.1}$$

where

$$\mu_k = \tfrac{2M_{kk}}{2M_{kk}-1} \qquad (k = 1, \ldots, n).$$

Comment. These formulae are not easy-to-derive ones. But once found, our task is greatly simplified because we are left with checking that Q_z given by (2.1) satisfies (1.3). $M_{k\bullet}$ denotes the kth row of M, and e_k stands for the kth column of the identity matrix I.

Proof. Given a $z \in \{-1,1\}^n$, define a matrix Q by

$$Q_{k\bullet} = \begin{cases} M_{k\bullet}T_z & \text{if } z_k = 1, \\ ((\mu_k - 1)M_{k\bullet} - \mu_k e_k^T)T_z & \text{if } z_k = -1 \end{cases} \qquad (k = 1, \ldots, n). \tag{2.2}$$

We shall prove that Q solves (1.3), which under the regularity assumption $M \geq I$ will mean that $Q = Q_z$. Let us note that this assumption implies that $2M_{kk} - 1 \geq 1$ and $\mu_k > 1$ for each k, and $M\Delta = M - I$. Take an arbitrary $k \in \{1, \ldots, n\}$. Now we have either $z_k = 1$, or $z_k = -1$.

If $z_k = 1$, then

$$|Q|_{k\bullet} = M_{k\bullet}, \tag{2.3}$$

hence

$$(Q - |Q|\Delta T_z)_{k\bullet} = M_{k\bullet}T_z - M_{k\bullet}\Delta T_z = M_{k\bullet}T_z - (M_{k\bullet} - e_k^T)T_z = e_k^T T_z = z_k I_{k\bullet} = I_{k\bullet},$$

so that

$$(Q - |Q|\Delta T_z)_{k\bullet} = I_{k\bullet}. \tag{2.4}$$

If $z_k = -1$, then (2.2) implies that

$$|Q_{kj}| = |(\mu_k - 1)M_{kj}z_j| = (\mu_k - 1)M_{kj} \tag{2.5}$$

for each $j \neq k$, and since

$$Q_{kk} = ((\mu_k - 1)M_{kk} - \mu_k)z_k = \left(\frac{M_{kk}}{2M_{kk}-1} - \frac{2M_{kk}}{2M_{kk}-1}\right)z_k = -\frac{M_{kk}}{2M_{kk}-1}z_k,$$

we have

$$|Q_{kk}| = \frac{M_{kk}}{2M_{kk}-1} = (\mu_k - 1)M_{kk}$$

which together with (2.5) gives

$$|Q|_{k\bullet} = (\mu_k - 1)M_{k\bullet}. \tag{2.6}$$

Now,

$$\begin{aligned}(Q-|Q|\Delta T_z)_{k\bullet} &= ((\mu_k-1)M_{k\bullet}-\mu_k e_k^T)T_z-(\mu_k-1)M_{k\bullet}\Delta T_z\\ &= ((\mu_k-1)M_{k\bullet}-\mu_k e_k^T)T_z-(\mu_k-1)(M_{k\bullet}-e_k^T)T_z\\ &= -e_k^T T_z = -z_k I_{k\bullet} = I_{k\bullet},\end{aligned}$$

hence (2.4) again holds. In this way we have shown that Q solves (1.3), and uniqueness of its solution implies that $Q=Q_z$. This proves that Q_z is given by (2.1). □

As a by-product of the proof we obtain an explicit description of the matrix $|Q_z|$.

Theorem 3. *Let $M\geq I$. Then for each $z\in\{-1,1\}^n$ the matrix $|Q_z|$ is given rowwise by*

$$|Q_z|_{k\bullet}=\begin{cases} M_{k\bullet} & \text{if } z_k=\ \ 1,\\ (\mu_k-1)M_{k\bullet} & \text{if } z_k=-1\end{cases}\qquad (k=1,\dots,n).$$

Proof. These are simply the equations (2.3) and (2.6). □

Next we show that both Q_z and $|Q_z|$ can be given by compact one-line formulae that, however, may be seen less transparent than the former ones.

Theorem 4. *Let $M\geq I$. Then for each $z\in\{-1,1\}^n$ the matrix Q_z and its absolute value are given by*

$$Q_z=\max\{T_zM,\, T_z((I-T_\mu)M+T_\mu)\}T_z \tag{2.7}$$

and

$$|Q_z|=\max\{T_zM,\, T_z(I-T_\mu)M\}, \tag{2.8}$$

where

$$\mu=\tfrac{2\mathrm{diag}(M)}{2\mathrm{diag}(M)-e}. \tag{2.9}$$

Comment. Observe closely the equation (2.7): first the entrywise maximum of two matrices is taken, then the result is postmultiplied by T_z. In (2.9) we use the Hadamard division of vectors so that

$$\mu_k=\tfrac{2M_{kk}}{2M_{kk}-1}\qquad (k=1,\dots,n)$$

as before (e is the vector of all ones).

Proof. For given $z\in\{-1,1\}^n$ set

$$Q=\max\{T_zM,\, T_z((I-T_\mu)M+T_\mu)\}T_z$$

and consider the difference

$$T_zM-T_z((I-T_\mu)M+T_\mu)=T_zT_\mu(M-I).$$

Because of $\mu>0$ and $M\geq I$ we have $T_\mu(M-I)\geq 0$, hence for each k there holds

$$(T_zM)_{k\bullet}\geq (T_z((I-T_\mu)M+T_\mu))_{k\bullet}$$

if $z_k=1$ and

$$(T_zM)_{k\bullet}\leq (T_z((I-T_\mu)M+T_\mu))_{k\bullet}$$

if $z_k = -1$. Thus

$$Q_{k\bullet} = (T_z M)_{k\bullet} T_z = M_{k\bullet} T_z = (Q_z)_{k\bullet}$$

if $z_k = 1$ and

$$Q_{k\bullet} = (T_z((I - T_\mu)M + T_\mu))_{k\bullet} T_z = ((\mu_k - 1)M_{k\bullet} - \mu_k e_k^T) T_z = (Q_z)_{k\bullet}$$

if $z_k = -1$, both by Theorem 2, hence $Q = Q_z$.

Similarly, for the matrix defined by

$$Q_a = \max\{T_z M, T_z(I - T_\mu)M\}$$

we have

$$T_z M - T_z(I - T_\mu)M = T_z T_\mu M$$

where $T_\mu M \geq 0$, hence

$$(Q_a)_{k\bullet} = M_{k\bullet} = |Q_z|_{k\bullet}$$

if $z_k = 1$ and

$$(Q_a)_{k\bullet} = (\mu_k - 1)M_{k\bullet} = |Q_z|_{k\bullet}$$

if $z_k = -1$, both by Theorem 3, which means that $Q_a = |Q_z|$. □

Finally, we bring about the utmost simplification.

Theorem 5. *Let $M \geq I$. Then for each $z \in \{-1, 1\}^n$ the matrix Q_z and its absolute value are given by*

$$Q_z = (D_z + T_z)M T_z - D_z T_z$$

and

$$|Q_z| = (D_z + T_z)M,$$

where

$$D_z = \tfrac{1}{2}(I - T_z)T_\mu$$

and

$$\mu = \tfrac{2\mathrm{diag}(M)}{2\mathrm{diag}(M) - e}.$$

Proof. For a $z \in \{-1, 1\}^n$, D_z is a diagonal matrix such that $(D_z)_{kk} = 0$ if $z_k = 1$ and $(D_z)_{kk} = \mu_k$ otherwise, therefore the matrix Q defined by

$$Q = (D_z + T_z)M T_z - D_z T_z$$

satisfies

$$Q_{k\bullet} = M_{k\bullet} T_z$$

if $z_k = 1$ and

$$Q_{k\bullet} = (\mu_k - 1)M_{k\bullet} T_z - \mu_k e_k^T T_z$$

if $z_k = -1$, hence $Q = Q_z$ by Theorem 2. The proof for $|Q_z|$ follows the same line. □

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