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A Hybrid Method for Solving Absolute Value Equations

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Technical report No. V-1223

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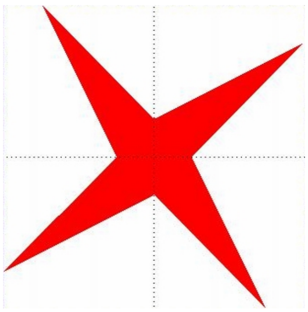
Abstract:

We present a hybrid method for solving an absolute value equation of the form $x + B|x| = b$ with $\varrho(|B|) < 1$. It first uses the iterative process $x^{i+1} = -B|x^i| + b$ performed until certain condition is met, then the unique solution x^* of the equation is found by solving a single system of linear equations. The method is shown to work whenever all entries of x^* are nonzero.[2]



Keywords:
Absolute value equation, iterative method, hybrid method.

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[2]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

1 Introduction

In [3], the authors proposed an iterative method for solving an absolute value equation of the form

$$x + B|x| = b. \tag{1.1}$$

They showed that if

$$\varrho(|B|) < 1, \tag{1.2}$$

then the sequence $\{x^i\}_{i=0}^{\infty}$ generated by

$$x^0 = b \tag{1.3}$$

and

$$x^{i+1} = -B|x^i| + b \qquad (i = 0, 1, \dots) \tag{1.4}$$

tends to the unique solution x^* of the equation (1.1) and, moreover, that there holds

$$|x^* - x^{i+1}| \le N|x^{i+1} - x^i| \tag{1.5}$$

for each $i \ge 0$, where

$$N = (I - |B|)^{-1} - I. \tag{1.6}$$

The condition (1.2) is equivalent to $N \ge 0$ (Horn and Johnson [2]).

In this note we show that under mild assumption (inequality (2.3) below) we can terminate generation of the sequence $\{x^i\}$ after a finite number of steps and use the information gathered in the last generated iteration to find the unique solution x^* by solving a single system of linear equations. This is what we call the hybrid method.

We use the following notation. Inequalities and absolute value are taken entrywise; "$\circ$" denotes the Hadamard (entrywise) product of vectors, $\mathrm{diag}(z)$ denotes the diagonal matrix with diagonal vector z and for $x \in \mathbb{R}^n$, the sign vector of x is defined by $(\mathrm{sgn}(x))_i = 1$ if $x_i \ge 0$ and $(\mathrm{sgn}(x))_i = -1$ otherwise ($i = 1, \dots, n$). Notice that $|x| = \mathrm{diag}(\mathrm{sgn}(x))x$ for each $x \in \mathbb{R}^n$. $\varrho(A)$ stands for the spectral radius of A and I is the identity matrix.

2 The hybrid method

We shall need the following auxiliary result.

Theorem 1. *If $x, y \in \mathbb{R}^n$ satisfy*

$$|x - y| < |y|, \tag{2.1}$$

then

$$0 < x \circ y < 2y \circ y. \tag{2.2}$$

Proof. For each i, (2.1) implies $y_i \ne 0$, and we have

$$|x_i y_i - y_i^2| = |x_i - y_i||y_i| < |y_i|^2 = y_i^2,$$

hence

$$-y_i^2 < x_i y_i - y_i^2 < y_i^2$$

and

$$0 < x_i y_i < 2y_i^2$$

which amounts to (2.2). □

Now the main idea behind the hybrid method is contained in the following theorem.

Theorem 2. *Let $\varrho(|B|) < 1$ and let the sequence $\{x^i\}$ generated by (1.3), (1.4) satisfy*

$$N|x^{i+1} - x^i| < |x^{i+1}| \tag{2.3}$$

for some i, where N is as in (1.6). Then the unique solution x^* of (1.1) is given by

$$x^* = (I + B\text{diag}(\text{sgn}(x^{i+1})))^{-1}b. \tag{2.4}$$

Proof. From (1.5) and (2.3) we have

$$|x^* - x^{i+1}| \leq N|x^{i+1} - x^i| < |x^{i+1}|,$$

hence $x^* \circ x^{i+1} > 0$ by Theorem 1 which means that both x^* and x^{i+1} belong to the interior of the same orthant of $\mathbb{R}^n$. Thus $\text{sgn}(x^*) = \text{sgn}(x^{i+1})$ and consequently $|x^*| = \text{diag}(\text{sgn}(x^*))x^* = \text{diag}(\text{sgn}(x^{i+1}))x^*$. Since x^* solves

$$x^* + B|x^*| = b,$$

it also solves

$$x^* + B\text{diag}(\text{sgn}(x^{i+1}))x^* = b,$$

hence x^* is given by the explicit formula (2.4). Invertibility of $I + B\text{diag}(\text{sgn}(x^{i+1}))$ is guaranteed by the assumption (1.2). □

Finally we show a necessary and sufficient condition for the hybrid method to work. Notice that $|x^*| > 0$ is equivalent to $x_i^* \neq 0$ for each i.

Theorem 3. *Let $\varrho(|B|) < 1$. Then the sequence $\{x^i\}$ generated by (1.3), (1.4) satisfies*

$$N|x^{i+1} - x^i| < |x^{i+1}| \tag{2.5}$$

for some i if and only if $|x^*| > 0$.

Proof. Let $|x^*| > 0$. Since $x_i \to x^*$, we have that

$$\lim_{i\to\infty} (|x^{i+1}| - N|x^{i+1} - x^i|) = |x^*| > 0,$$

hence by the definition of limit there exists an i_0 such that

$$|x^{i+1}| - N|x^{i+1} - x^i| > 0$$

holds even for each $i \geq i_0$. Conversely, if (2.5) holds for some i, then as in the proof of Theorem 2 we obtain

$$|x^* - x^{i+1}| < |x^{i+1}|$$

which means that $|x^*| > 0$ since $x_j^* = 0$ for some j would imply $|x_j^{i+1}| < |x_j^{i+1}|$, a contradiction. □

Bibliography

[1] W. Barth and E. Nuding, *Optimale Lösung von Intervallgleichungssystemen*, Computing, 12 (1974), pp. 117–125. 1

[2] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985. 1

[3] J. Rohn, V. Hooshyarbakhsh, and R. Farhadsefat, *An iterative method for solving absolute value equations and sufficient conditions for unique solvability*, Optimization Letters, 8 (2014), pp. 35–44. 1