Institute of Computer Science
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Report on the Last Work by Dr. Erich Nuding

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Technical report No. V-1235

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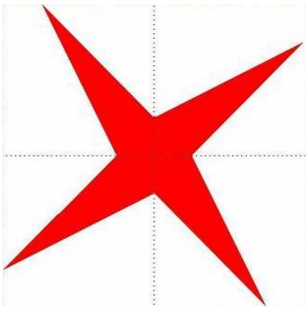
Abstract:

This is a facsimile copy of a 1994 report on the unpublished last paper by Dr. Erich Nuding. It is being made public here in the hope that even after twenty-two years it may be of interest for researchers working in the area of interval computations because of the intriguing concept of the "fourth modality" which has not been rediscovered during a quarter of century which has elapsed since its original formulation.[2]



Keywords:
Set-valued mapping, interval linear equations, solution set, fourth modality.

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[2]Below: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding 1974).

Report on the last work by Dr. Erich Nuding

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1. Origin of this report. Dr. Erich Nuding died in November 1991. Shortly afterwards I was asked by his son, Herr G. Nuding, and Frau Dr. Kahlert-Warmbold to examine his posthumous papers and to give my evaluation of the results contained therein. This report, which covers approximately a half of the work, handles the results given in the last paper by Dr. Nuding, completed shortly before his death.

2. The material. I received a copy of the manuscript in German, typed by Dr. Kahlert-Warmbold and entitled ''Abschrift des Manuskriptes von Herrn Dr. E. Nuding'' (69 pages + 8 pages of amendments), and under a separate cover a ''Kommentar zur letzten Arbeit von Herrn Dr. Nuding'' which consists of parts of correspondence between Dr. Nuding and Dr. Kahlert-Warmbold related to the first part of the manuscript (5 p.), references (1 p.) and very detailed contents of the manuscript with cross-references (12 p.). Additionally I was sent a proof of one theorem (2 p.). With this single exception, all the results are contained in the ''Abschrift ...'' to which I will refer in the sequel.

3. Timing and circumstances. The manuscript, which was seemingly motivated by an attempt to put together previous results in a unified approach, was started on August 3, 1991 and finished on October 31, 1991; Dr. Nuding died less than a month later. On two places he hinted indirectly that he was competing with time (''...es mir

immer schwerer fällt...'', p. 1; ''...in Anbetracht der Zeit...'', p. 55).

4. The contents. Dr. Nuding starts in section 1 with basic notations on set-valued mappings (''mengenwertige Abbildungen'')

$$F: M \rightarrow P(M') \qquad (1)$$

where it is clear from the context (although not stated explicitly) that $P(M')$ is the set of all subsets of M'. In the first two sections he considers in parallel a mapping

$$\phi : P(M) \rightarrow P(M')$$

with the property

$$X \subset X' \Rightarrow \phi(X) \subset \phi(X')$$

but this parallelism is sometimes confusing and the author omitted it from the beginning of section 3. In section 2, which is devoted to the concept of solution of an equation of the type $F(x) = Y$, where $Y \in P(M')$, he defines two concepts of a solution:

$$\mathrm{ext}(F,Y) = \{x \in M;\ F(x) \cap Y \neq \emptyset\} \qquad (2)$$

called an ''outer solution'' (''äussere Lösung'') or ''solution set'' (''Lösungsmenge'') and

$$\mathrm{int}(F,Y) = \{x \in M;\ F(x) \subset Y\} \qquad (3)$$

called an ''inner solution'' (''innere Lösung''). Both these solutions are evidently taken from the interval analysis where they were developed in 1960's and 1970's. To my knowledge, the concept of inner solutions was introduced by Dr. Nuding in his well-known paper with Wilhelm [1]. The ''inner solutions'' are now preferrably called ''tolerance solutions'' and have been studied since by various authors (Neumaier [2], Kelling [3], Shary [4], myself [5]), but the priority of introducing this concept is due to Dr. Nuding.

The introduction of the solution set (2) enables the author to introduce an ''inverse set-valued mapping''

$$F^{-1}: M' \rightarrow P(M)$$

by

$$F^{-1}(y) = \text{ext}(F,\{y\})$$

for $y \in M'$. This is then employed in the definition of an interesting concept of the ''fourth modality'' (''vierte Modalität'') defined (p. 15) as the set

$$\text{int}(F^{-1},X) \qquad (4)$$

where ''int...'' is given by (3). With exception of two historical remarks on pp. 16 and 24, which are rather vague, the author did not elaborate on the concept and mentioned it only once more, on the last page (amendment A8). In the manuscript he left unanswered the natural question, why the ''fourth'' modality; in his reply to this answer put by Dr. Kahlert-Warmbold he wrote: ''Ich meine ja, dass die Philosophen seit Aristoteles die 4. Modalität übersehen haben könnten. Selbst bei Kant sind es immer deren 3! Und die liegen schief.''. I will devote this concept a special paragraph later.

From section 3 on, Dr. Nuding tries to handle systematically some basic problems concerning set-valued mappings using the operators ext(F,Y) and int(F,Y). In section 3 he defines lower and upper semi-continuity of set-valued mappings (p. 30) and gives a number of equivalent formulations. In section 3a he considers a special case of interval functions (in semiordered vector spaces) and investigates the connection of semicontinuity of these functions in the above sense with the classical semicontinuity of the functions inf F, sup F which form the bounds of the interval function. Another connections, further topological properties and the behaviour of F(X) on the boundary of X are examined in sections 3b, 3c and 3d. A very short section 4 ''Grenzwerte'' (1 p.) brings definitions of $\underline{\lim}\, X_n$ and $\overline{\lim}\, X_n$ for a sequence of sets $\{X_n\}$, which is seemingly due to Hausdorff. These concepts are utilized in section 5 (''Approximations''). Whereas the previous sections brought only a few proofs, here the Theorem 60 is given with a detailed proof, and both the theorem and the proof seem nontrivial to me. The last section 6 handles convex sets and concave mappings. Convexity of a set-valued mapping is defined on p. 62. Dr. Nuding especially emphasizes Theorem 75 on p. 65 which demonstrates a nontrivial ''concavity property'' of ext(F,y). The result sent to me separately later (cf. part 2 of this report) is also related to this theorem. The last 8 pages of amendments bring some improvements and extensions.

5. Evaluation. I have read the manuscript with interest, yet I do not feel competent to express a qualified opinion about the matter of sections 3-6. I think they might be novel, as they are based on the concepts of the sets (2) and (3) which I have never seen to be considered in this general setting. But as I have never worked in topology or related areas, I cannot be sure of it; therefore I would propose to let the manuscript to be examined by someone who is better oriented in these areas.

On the other side, I am sure that the concept of the ''fourth modality'' is new and I found it interesting and posing nontrivial questions. I will handle this matter in its interval analytic setting in the next part, adding some my own ideas on the subject.

6. ''Fourth modality''. Although Dr. Nuding did not explain what were the first three ''modalities'', it is interesting that in my view it is also the fourth concept, yet seemingly in another counting. Consider an $m \times n$ interval matrix

$$A^I = \{A;\ A_c - \Delta \leq A \leq A_c + \Delta\}$$

and an m-dimensional interval vector

$$b^I = \{b;\ b_c - \delta \leq b \leq b_c + \delta\}.$$

With A^I fixed, consider a set-valued mapping

$$F(x) = A^I x = \{Ax;\ A \in A^I\}. \qquad (5)$$

It follows from the well-known theorem by Oettli and Prager [6] that

$$F(x) = [A_c x - \Delta|x|,\ A_c x + \Delta|x|].$$

Then for the set

$$X(A^I, b^I) = \{x;\ Ax = b \text{ for some } A \in A^I,\ b \in b^I\},$$

which is usually called the ''solution set'' in interval analysis, we have

$$X(A^I, b^I) = \{x;\ F(x) \cap b^I \neq \emptyset\} = \exp(F, b^I).$$

Next, for the set of tolerance solutions

$$X_t(A^I, b^I) = \{x;\ Ax \in b^I \text{ for each } A \in A^I\}$$

we have

$$X_t(A^I, b^I) = \{x;\ F(x) \subset b^I\} = \mathrm{int}(F, b^I),$$

hence the two concepts of solution introduced by Dr. Nuding correspond exactly to those already known. Recently, Shary [7] introduced the third concept, called by him the ''controlled solution set'':

$$X_c(A^I, b^I) = \{x;\ b^I \subset F(x)\}.$$

It is worth mentioning that these three sets, although essentially different, share very similar descriptions:

$$X(A^I, b^I) = \{x;\ |A_c x - b_c| \leq \Delta|x| + \delta\},$$
$$X_t(A^I, b^I) = \{x;\ |A_c x - b_c| \leq -\Delta|x| + \delta\},$$
$$X_c(A^I, b^I) = \{x;\ |A_c x - b_c| \leq \Delta|x| - \delta\}$$

(cf. [6], [5], [8]). It can be shown that X_t is a convex polytope [5] whereas X and X_c are generally nonconvex.

Now, Dr. Nuding's ''fourth modality'' (4) for the mapping F given by (5) is simply the set

$$X_f(A^I, x^I) = \mathrm{int}(F^{-1}, x^I) = \{b;\ X(A^I, \{b\}) \subset x^I\},$$

where an interval vector x^I is assumed to be given. Hence, in contrast to the previous three cases, where we were looking for some solution x, here we look for a right-hand side vector b for which the solution set $X(A^I, \{b\})$ would be completely contained within a prescribed range x^I. To my knowledge, nobody has introduced or studied such a concept so far, so that Dr. Nuding's view is new. Unfortunately, as I mentioned above, Dr. Nuding did not elaborate on the idea. In what follows, I shall shortly describe some of my own ideas on this matter for the special (but mostly studied) case $m = n$, i.e. when A^I is square.

Let us first recall a notation used in [9]. By

$$Y = \{y \in R^n;\ |y_j| = 1 \text{ for each } j \in \{1,\dots,n\}\}$$

we introduce the set of all ±1-vectors in R^n. For each $y \in Y$, let

$$T_y = \operatorname{diag}\{y_1,\dots,y_n\}$$

be the diagonal matrix with diagonal vector y. Assuming as above that A^I is of the form

$$A^I = [A_c - \Delta,\ A_c + \Delta],$$

we introduce, for each pair y,z of vectors from Y, the matrix

$$A_{yz} = A_c - T_y \Delta T_z.$$

Obviously, $A_{yz} \in A^I$ for each $y,z \in Y$. Then we have this characterization:

Theorem 1. Let A^I be regular and $x^I = [\underline{x}, \overline{x}]$. Then $b \in X_f(A^I, x^I)$ if and only if

$$\underline{x} \leq A_{yz}^{-1} b \leq \overline{x} \qquad (6)$$

holds for each $y,z \in Y$.

Proof. According to the ''convex-hull theorem'' 2.2 in [9], we have

$$\operatorname{Conv} X(A^I, \{b\}) = \operatorname{Conv} \{A_{yz}^{-1} b;\ y,z \in Y\}.$$

Hence, $X(A^I, \{b\}) \subset x^I$ if and only if $\operatorname{Conv} X(A^I, \{b\}) \subset x^I$, which is equivalent to $A_{yz}^{-1} b \in x^I$ for each $y,z \in Y$.

Theorem 1 enables us, at least theoretically, to check by finite means whether a given right-hand side b belongs to X_f. This way, of course, is very impractical. Nevertheless, it implies this result:

Theorem 2. For A^I regular and arbitrary x^I, the set $X_f(A^I,x^I)$ is a convex polyhedron.

Proof. $b \in X_f$ if and only if it satisfies all the systems (6) for all $y,z \in Y$. Hence, the set X_f is described by a system of linear inequalities and therefore it is a convex polytope. Since $X_f(A^I,x^I) \subset \{Ax;\ A \in A^I,\ x \in x^I\}$, we can see that X_f is bounded, so that it is a convex polyhedron.

Despite these two results, some problems remain open:

Open problems.

1 Does there exist a simpler description of X_f ?

2 Does there exist a verifiable necessary and sufficient condition for $X_f \neq \emptyset$?

3 Does there exist a polynomial-time algorithm for finding a $b \in X_f$ provided $X_f \neq \emptyset$?

4 What are the vertices of X_f ?

These problems show that Dr. Nuding's idea of the ''fourth modality'' is fruitful. I do hope that the results will sometimes be further developed.

7. Acknowledgment. I wish to highly acknowledge the work done by Dr. Kahlert-Warmbold who undertook the difficult task of typing the handwritten manuscript, thereby making it accessible.

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