Institute of Computer Science
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Interval Matrices: Regularity Yields Singularity

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Technical report No. V-1239

27.10.2016

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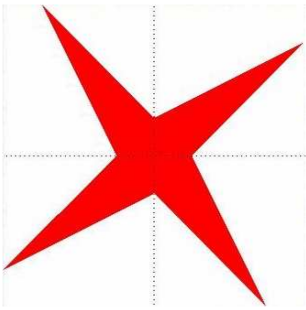
Abstract:

It is proved that regularity of an interval matrix implies singularity of two related interval matrices.[2]



Keywords:
Interval matrix, regularity, singularity.

[1]This work was supported with institutional support RVO:67985807.

[2]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

1 Introduction

A square interval matrix

$$[A - D, \, A + D] = \{ \, B \mid A - D \le B \le A + D \, \}$$

is called regular if each $B \in [A - D, \, A + D]$ is nonsingular, and is said to be singular otherwise. In this report we show that regularity of $[A - D, \, A + D]$ implies singularity of two related interval matrices. This is an atypical result which, in this author's knowledge, bears no analogy in literature.

2 Main result

Theorem 1. *If $[A - D, \, A + D]$ is regular, then both the interval matrices*

$$[A^{-1}D - I, \, A^{-1}D + I], \tag{2.1}$$

$$[DA^{-1} - I, \, DA^{-1} + I] \tag{2.2}$$

are singular (I is the identity matrix).

Proof. Let $[A - D, \, A + D]$ be regular. Put $C = A^{-1}D$. Notice that $I - C = A^{-1}(A - D)$, as a product of two nonsingular matrices, is nonsingular. Consider the matrix

$$(I - C)^{-1}(I + C) = (A - D)^{-1}(A + D).$$

By [3, Thm. 1.2] the matrix $(A - D)^{-1}(A + D)$ is a P-matrix, hence so is $(I - C)^{-1}(I + C)$, and a theorem by Gale and Nikaido [2] implies existence of an $\tilde{x} > 0$ satisfying

$$(I - C)^{-1}(I + C)\tilde{x} > 0. \tag{2.3}$$

Set $x = (I - C)^{-1}\tilde{x}$. Then $(I - C)x = \tilde{x} > 0$, hence

$$Cx < x, \tag{2.4}$$

and from (2.3) we have

$$\begin{aligned} 0 < (I - C)^{-1}(I + C)\tilde{x} &= (I - C)^{-1}(I + C)(I - C)x \\ &= (I - C)^{-1}(I - C^2)x \\ &= (I - C)^{-1}(I - C)(I + C)x \\ &= (I + C)x, \end{aligned}$$

which gives $-x < Cx$ and together with (2.4)

$$-x < Cx < x,$$

which is

$$|Cx| < x. \tag{2.5}$$

This inequality shows that $x > 0$. Now define

$$S = C - \operatorname{diag}(y),$$

where $y = (y_i)$ is given by

$$y_i = (Cx)_i / x_i \quad (i = 1, \ldots, n),$$

then $|S - C| \leq I$ due to (2.5) and $(Sx)_i = (Cx)_i - y_i x_i = 0$ for each i, hence $Sx = 0$ and S is a singular matrix in (2.1).

Next, regularity of $[A - D,\, A + D]$ implies that of its transpose $[A^T - D^T,\, A^T + D^T] = \{\, B^T \mid B \in [A - D,\, A + D]\,\}$ which according to what has just been proved yields singularity of $[(A^T)^{-1}D^T - I,\, (A^T)^{-1}D^T + I] = [(DA^{-1})^T - I,\, (DA^{-1})^T + I]$ and thereby also that of its transpose (2.2). □

3 Consequence

As a consequence we obtain the following purely linear algebraic result.

Theorem 2. *Let A be invertible. Then there exists a singular matrix S satisfying either*

$$|A - S| \leq I,$$

or

$$|A^{-1} - S| \leq I.$$

Proof. Consider the interval matrix $[A - I,\, A + I]$. If it is singular, then we are done; if it is regular, then $[A^{-1} - I,\, A^{-1} + I]$ is singular by Theorem 1. □

In other words, *either A or A^{-1} can be brought to a singular matrix by shifting diagonal entries by componentwise magnitudes of at most 1.*

Bibliography

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