Institute of Computer Science
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Overdetermined Absolute Value Equations

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Technical report No. V-1265

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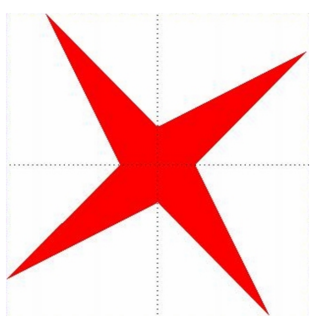
Abstract:

We consider existence, uniqueness and computation of a solution of an absolute value equation in the overdetermined case.[2]



Keywords:
Absolute value equations, overdetermined system.

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[2]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

1 Introduction

The absolute value equation

$$Ax + B|x| = b \tag{1.1}$$

has been studied so far for the square case only ($A, B \in \mathbb{R}^{n\times n}$). In this report we consider the rectangular case ($A, B \in \mathbb{R}^{m\times n}$); the assumption (2.1) made below ensures that $m \geq n$, so that in fact we investigate the overdetermined case only.

Notation used: $|x|$ is the entrywise absolute value of x, ϱ denotes the spectral radius, I is the identity matrix and $A^\dagger$ stands for the Moore-Penrose inverse of A.

2 The result

We shall handle the questions of existence, uniqueness and computation of a solution in frame of a single theorem.

Theorem 1. *Let $A, B \in \mathbb{R}^{m\times n}$ satisfy*

$$\operatorname{rank}(A) = n \tag{2.1}$$

and

$$\varrho(|A^\dagger B|) < 1. \tag{2.2}$$

Then for each $b \in \mathbb{R}^m$ the sequence $\{x^i\}_{i=0}^\infty$ generated by

$$x^0 = A^\dagger b, \tag{2.3}$$

$$x^{i+1} = -A^\dagger B|x^i| + A^\dagger b \qquad (i = 0, 1, 2, \dots) \tag{2.4}$$

tends to a limit x^*, and we have:

(i) if $Ax^* + B|x^*| = b$, then x^* is the unique solution of (1.1),

(ii) if $Ax^* + B|x^*| \neq b$, then (1.1) possesses no solution.

Proof. For clarity, we divide the proof into several steps.

(a) From (2.4) we have

$$|x^{i+1} - x^i| \leq |A^\dagger B||x^i - x^{i-1}|$$

for each $i \geq 1$ and since $|A^\dagger B|^j \to 0$ as $j \to \infty$ due to (2.2), proceeding as in the proof of Theorem 1 in [2] we prove that $\{x^i\}$ is a Cauchian sequence, thus it is convergent, $x^i \to x^*$. Taking the limit in (2.4) we obtain that $x^* = -A^\dagger B|x^*| + A^\dagger b$, i.e., x^* solves the equation

$$x + A^\dagger B|x| = A^\dagger b. \tag{2.5}$$

(b) Assume that $\tilde{x}$ also solves (2.5). Then

$$|x^* - \tilde{x}| \leq |A^\dagger B||x^* - \tilde{x}|,$$

hence

$$(I - |A^\dagger B|)|x^* - \tilde{x}| \leq 0$$

and premultiplying this inequality by the inverse of $I - |A^{\dagger}B|$ which is nonnegative due to (2.2) results in

$$|x^* - \tilde{x}| \leq 0,$$

hence $x^* = \tilde{x}$ which means that x^* is the unique solution to (2.5).

(c) We prove that if x solves (1.1), then $x = x^*$. Indeed, in that case it also solves the preconditioned equation

$$A^{\dagger}Ax + A^{\dagger}B|x| = A^{\dagger}b \tag{2.6}$$

and since $A^{\dagger} = (A^TA)^{-1}A^T$ due to (2.1), $A^{\dagger}A = I$ and x solves (2.5) so that $x = x^*$.

(d) If $Ax^* + B|x^*| = b$, then x^* is a solution of (1.1) and it is unique by (c).

(e) If $Ax^* + B|x^*| \neq b$, then existence of a solution x to (1.1) would mean that $x = x^*$ by (c), hence $Ax^* + B|x^*| = b$, a contradiction. □

We have this immediate consequence.

Theorem 2. *Under conditions (2.1) and (2.2) the equation (1.1) possesses for each* $b \in \mathbb{R}^m$ *at most one solution.*

Bibliography

[1] W. Barth and E. Nuding, *Optimale Lösung von Intervallgleichungssystemen*, Computing, 12 (1974), pp. 117–125.

[2] J. Rohn, V. Hooshyarbakhsh, and R. Farhadsefat, *An iterative method for solving absolute value equations and sufficient conditions for unique solvability*, Optimization Letters, 8 (2014), pp. 35–44.