Institute of Computer Science
Academy of Sciences of the Czech Republic

Absolute Value Mapping

Jiří Rohn
http://uivtx.cs.cas.cz/∼rohn

Technical report No. V-1266

05.05.2019

Pod Vodárenskou věží 2, 182 07 Prague 8, phone: +420 266 051 111, fax: +420 286 585 789,
e-mail:rohn@cs.cas.cz

Institute of Computer Science
Academy of Sciences of the Czech Republic

Absolute Value Mapping

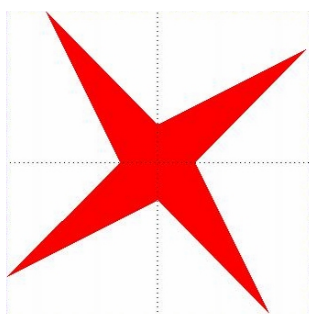
Jiří Rohn[1]
http://uivtx.cs.cas.cz/∼rohn

Technical report No. V-1266

05.05.2019

Abstract:

We prove a necessary and sufficient condition for an absolute value mapping to be bijective. This result simultaneously gives a characterization of unique solvability of an absolute value equation for each right-hand side.[2]



Keywords:
Absolute value mapping, bijectivity, interval matrix, regularity, absolute value equation, unique solvability.

[1]This work was supported with institutional support RVO:67985807.

[2]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

1 Introduction

The mapping

$$f_{AB}(x) = Ax + B|x|, \tag{1.1}$$

where A, $B \in \mathbb{R}^{n\times n}$, is called an absolute value mapping (the absolute value of a vector is understood entrywise). In this report we are solely interested in condition under which f_{AB} is bijective, i.e., is a one-to-one mapping of $\mathbb{R}^n$ onto itself. We show below that the problem is closely connected with regularity of interval matrices.

Our result given in Theorem 3 can be also seen as a necessary and sufficient condition for unique solvability of an absolute value equation

$$Ax + B|x| = b$$

for each right-hand side $b \in \mathbb{R}^n$, a property for which only sufficient conditions have been known so far.

2 Auxiliary results

For the proof of the main theorem we shall need two auxiliary results that are of independent interest. Let us recall that a square matrix is called a P-matrix if all its principal minors are positive. The first result is due to Murty [2, Thm. 4.2]; x^+ and x^- are defined by $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$ (entrywise).

Theorem 1. *Let $C \in \mathbb{R}^{n\times n}$. Then the mapping*

$$g_C(x) = x^+ - Cx^-$$

is a bijection of $\mathbb{R}^n$ onto itself if and only if C is a P-matrix.

The second result is due to Rump [4, Thm. 4.1]. The set of the form

$$[F - G, F + G] = \{ H \mid F - G \le H \le F + G \}$$

where $F, G \in \mathbb{R}^{n\times n}$, $G \ge 0$, is called an interval matrix and it is said to be regular if each matrix H contained therein is nonsigular.

Theorem 2. *Let $C - I$ be nonsingular. Then C is a P-matrix if and only if the interval matrix*

$$[(C - I)^{-1}(C + I) - I,\ (C - I)^{-1}(C + I) + I]$$

is regular.

In the original Rump's formulation nonsingularity of both $C - I$ and $C + I$ was assumed; it was shown later in [3, Thm 2] that the second assumption is superfluous.

3 Characterization

Assume that $A+B$ is nonsingular; then we can define the matrix

$$C = (A+B)^{-1}(A-B)$$

which satisfies

$$\begin{aligned} C - I &= (A+B)^{-1}(A-B) - (A+B)^{-1}(A+B) = -2(A+B)^{-1}B, \\ C + I &= (A+B)^{-1}(A-B) + (A+B)^{-1}(A+B) = \ \ 2(A+B)^{-1}A, \end{aligned}$$

and $C-I$ becomes nonsingular under an additional assumption of nonsingularity of B. Then we can introduce a matrix D by

$$D = (C-I)^{-1}(C+I) = -B^{-1}(A+B)(A+B)^{-1}A = -B^{-1}A.$$

Theorem 3. *Let both B and $A+B$ be nonsingular. Then the mapping (1.1) is a bijection of $\mathbb{R}^n$ onto itself if and only if the interval matrix*

$$[D-I,\ D+I]$$

is regular.

Proof. Because x and $|x|$ can be decomposed as $x = x^+ - x^-$ and $|x| = x^+ + x^-$, we have

$$\begin{aligned} f_{AB}(x) &= A(x^+ - x^-) + B(x^+ + x^-) = (A+B)x^+ - (A-B)x^- \\ &= (A+B)(x^+ - Cx^-) = (A+B)g_C(x) \end{aligned}$$

and since $A+B$ is nonsingular, f_{AB} is a bijection of $\mathbb{R}^n$ onto itself if and only if g_C possesses the same property which by Theorem 1 is the case if and only if C is a P-matrix. Now, by Theorem 2, C is a P-matrix if and only if the interval matrix

$$[D-I, D+I]$$

is regular which concludes the proof. $\square$

4 Checking

Thus checking bijectivity of f_{AB} may be performed by the following MATLAB file whose subroutine can be downloaded from `http://uivtx.cs.cas.cz/~rohn/other/regising.m`.

```
function b=bijectivity(A,B)
%
% b== 1: the mapping x --> A*x + B*abs(x) is bijective,
% b==-1: the mapping is not bijective.
%
n=size(A,1); I=eye(n,n);
if rank(B)<n || rank(A+B)<n
    error('Condition not satisfied.')
end
D=-inv(B)*A;
S=regising(D,I);
if isempty(S), b=1; else b=-1; end
```

Bibliography

[1] W. Barth and E. Nuding, *Optimale Lösung von Intervallgleichungssystemen*, Computing, 12 (1974), pp. 117–125.

[2] K. G. Murty, *On the number of solutions to the complementarity problem and spanning properties of complementary cones*, Linear Algebra and Its Applications, 5 (1972), pp. 65–108.

[3] J. Rohn, *On Rump's characterization of P-matrices*, Optimization Letters, 6 (2012), pp. 1017–1020.

[4] S. M. Rump, *On P-matrices*, Linear Algebra and Its Applications, 363 (2003), pp. 237–250.